Physics

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Preface

I am a student pursuing a bachelor's degree in mathematics and master's degree in computer science at Stanford University. I took mechanics and electromagnetism in my freshman year but quickly got bored as the course was taught in a way that forced us to memorize the formulas and results without actually explaining the physics.

In my junior year I learned about field theory and started to appreciate physics. Here is a list of courses I have taken since then.

- Classical Electrodynamics, with Prof. Srinivas Raghu, Winter 2023
- Observational Astrophysics, with Prof. Steven Allen, Spring 2023
- Statistical Mechanics, with Prof. Stephen Shenker, Fall 2023,
- Experimental Physics, with Prof. David Goldhaber-Gordon, Spring 2024
- Quantum Field Theory, with Prof. Michael Peskin, Fall 2024
- General Relativity, with Prof. Roger Blandford, Winter 2025

Here I aim to summarize what I have learned so far about theoretical physics in a small pamphlet. The mathematics requirement is a working knowledge of algebra and calculus. The physics ideas shall be self-contained. This is a chance for me to reflect on the physical principles that have been recurring across all natural phenomenon, and, for the readers, this little book serves as an introductory guide to get an overview of physics with minimal time and effort.

This project, however, just like physics, is not complete, and may never be completed.

Contents

Part I

Force

Chapter 1

Laws of Motion

1.1 Early Developments

- Aristotle (384–322 BCE): Proposed that objects have a natural place and that motion requires a force. Aristotle's theories on natural motion, the four elements, and the geocentric model of the universe dominated Western thought for nearly two millennia.
- Archimedes (c. 287–212 BCE): Developed the principles of levers and buoyancy and made significant advancements in geometry and hydrostatics.
  - **Archimedes' Principle:** A body submerged in a fluid experiences a buoyant force equal to the weight of the fluid it displaces.

    $$F_b = \rho_{\text{fluid}} V g$$

    where F_b is the buoyant force, ρ_{fluid} is the density of the fluid, V is the volume of the fluid displaced, and g is the acceleration due to gravity.
  - **Principle of the Lever:** A lever is in static equilibrium when the sum of the torques is zero.

    $$F_1 d_1 = F_2 d_2$$

    where F_1 and F_2 are the forces applied at distances d_1 and d_2 from the fulcrum, respectively.
- Galileo Galilei (1564–1642): Conducted experiments on the motion of objects and concluded that objects continue in a state of motion unless acted upon by an external force.
  - **Law of Inertia:** Galileo suggested that an object in motion continues in a state of uniform motion unless acted upon by an external force, a precursor to Newton's First Law.
  - **Law of Uniform Acceleration:** Galileo discovered that the distance covered by a uniformly accelerating object is proportional to the square of the time elapsed. For a body in free fall,

    $$s = \frac{1}{2} g t^2$$

where s is the distance fallen, g is the acceleration due to gravity, and t is the time elapsed.

- **Equations of Motion:** Galileo also described uniformly accelerated motion with the following:

$$v = u + at$$
$$s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$

where u is the initial velocity, v is the final velocity, a is the acceleration, t is the time, and s is the distance.

- René Descartes (1596–1650): Descartes developed early versions of the laws of motion and the concept of the conservation of momentum, as well as the Cartesian coordinate system.
  - **Law of Inertia:** Descartes' principle that an object continues in its state of rest or uniform motion unless acted upon by an external force can be written as:
$$\frac{d\mathbf{p}}{dt} = 0 \quad \text{(in the absence of external forces)}$$
where $\mathbf{p}$ is the momentum.
  - **Conservation of Momentum:** Descartes' understanding of the conservation of momentum, particularly in elastic collisions, can be expressed as:
$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$$
where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta of two objects before collision and $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are their momenta after collision.

1.2 Newton's Law of Motion

Theorem 1 (Newton's Laws of Motion)**.** *Newton's Laws of Motion consist of three fundamental principles:*

- ***First Law:*** *An object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an unbalanced force.*
- ***Second Law:*** *The net force acting on an object is equal to the mass of the object multiplied by its acceleration. Mathematically, this is expressed as*
$$\mathbf{F} = m\mathbf{a}$$
where $\mathbf{F}$ *is the net force,* m *is the mass, and* $\mathbf{a}$ *is the acceleration.*
- ***Third Law:*** *For every action, there is an equal and opposite reaction. If object A exerts a force* $\mathbf{F}$ *on object B, then object B exerts a force* $-\mathbf{F}$ *on object A. In other words,*
$$\mathbf{F}_{A \text{ on } B} = -\mathbf{F}_{B \text{ on } A}$$

Example 2 (Conservation of Energy)**.**

$$\int F(x)\,dx = m\int a(x)\,dx = m\int \frac{d^2x}{dt^2}\frac{dx}{dt}\,dt = \tfrac{m}{2}\int \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 dt = \tfrac{1}{2}m\big(v_1^2 - v_0^2\big).$$

Example 3 (Elastic Collisions)**.** *The collision is most simply described in a frame of reference in which the centre of mass of the two particles is at rest. The velocities in the center of mass frame of the particles before the collision are related to their velocities* v_1 *and* v_2 *in the laboratory system by*

$$v_{1o} = \frac{m_2 v}{m_1 + m_2}, \quad v_{2o} = -\frac{m_1 v}{m_1 + m_2}$$

where $v = v_1 - v_2$*.*

Because of the law of conservation of momentum, the momenta of the two particles remain equal and opposite after the collision, and are also unchanged in magnitude, by the law of conservation of energy. Thus, in the C system the collision simply rotates the velocities, which remain opposite in direction and unchanged in magnitude.

Chapter 2

Law of Universal Gravitation

2.1 Early Developments

- Ptolemy (c. 100–170 CE)'s work, the Almagest, formulated the Ptolemaic system of astronomy, which described a geocentric universe with complex epicycles.
- Nicholas Copernicus (1473–1543) proposed the heliocentric model of the universe, which placed the Sun at the center rather than the Earth.

Theorem 4 (Kepler's Laws of Planetary Motion, 1609)**.** *Using observational data provided by Tycho Brahe, Johannes Kepler formulated his three laws of planetary motion, which described the orbits of planets around the Sun:*

1. ***Law of Orbits:*** *Each planet moves in an elliptical orbit with the Sun at one of the two foci. Mathematically, this can be described by the equation of an ellipse:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is the semi-major axis and b is the semi-minor axis.

2. ***Law of Areas:*** *The line segment joining a planet and the Sun sweeps out equal areas A in equal intervals of time t. This can be expressed as:*

$$\frac{dA}{dt} = constant$$

3. ***Law of Periods:*** *The square of the orbital period T of a planet is directly proportional to the cube of the semi-major axis a of its orbit:*

$$T^2 \propto a^3$$

2.2 Universal Gravitation

Before Newton, the movement of celestial bodies (such as planets and stars) and terrestrial objects (such as falling apples and projectiles) were believed to be governed by distinct sets of principles. Newton demonstrated that the laws of physics are universal, applying consistently to both celestial and terrestrial phenomena. At the time, Newton also developed calculus in order to do calculations.

Theorem 5 (Newton's Law of Universal Gravitation)**.** *Newton's law of universal gravitation states that every point mass attracts every other point mass in the universe with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Mathematically, the force* $\mathbf{F}$ *between two masses* m_1 *and* m_2*, separated by a distance* r*, is given by:*

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

where:

- G *is the gravitational constant, approximately equal to* $6.674 \times 10^{-11}\ Nm^2kg^{-2}$*,*
- $\hat{\mathbf{r}}$ *is the unit vector in the direction from one mass to the other.*

Remark 6. *Johannes Kepler's laws of planetary motion could be derived from Newton's law of universal gravitation.*

Proof. TODO □

Example 7 (Brachistochrone curve (curve of fastest descent), 1696)**.** *Given two points* A *and* B *in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at* A *and reaches* B *in the shortest time?*

By conservation of energy, we see that velocity only depends on the vertical length of the descent h*:*

$$v(h) = \sqrt{2gh}.$$

Note that fastest descent is a local property. Discretize the curve as n *segments* $(x_1, y_1), \ldots, (x_n, y_n)$*, and the velocity* v_i *in each segment* (x_i, y_i) *is constant.*

Now we only analyze the transition from (x_1, y_1) *to* (x_2, y_2)*. Because of the fastest descent property,*

$$\frac{\sqrt{x_1^2 + y_1^2}}{v_1} + \frac{\sqrt{x_2^2 + y_2^2}}{v_2}$$

is minimized. Taking derivatives, we find Snell's law:

$$\frac{v_2}{v_1} = \frac{y_2/\sqrt{x_2^2 + y_2^2}}{y_1/\sqrt{x_1^2 + y_1^2}} = \frac{\sin\theta_2}{\sin\theta_1}.$$

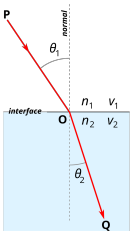

Figure 2.1: Law of Refraction

Thus,

$$\frac{dx}{\sqrt{dx^2 + dy^2}}\sin\theta(h) = \sqrt{2gh} \cdot constant.$$

Solving gives

$$\frac{dx}{dy} = \sqrt{\frac{y}{2a - y}}, \quad x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

Chapter 3

Mechanics

3.1 Lagrangian and Hamiltonian

Theorem 8 (principle of stationary action)**.** *Every mechanical system is characterized by a definite function $L(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s, t)$, or briefly $L(q, \dot{q}, t)$, and the motion of the system is such that a certain condition is satisfied.*

Let the system occupy, at the instants t_1 and t_2, positions defined by two sets of values of the coordinates, $q_1, q_2, \ldots, q_s$. Then the condition is that the system moves between these positions in such a way that the integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

takes a stationary value. In other words,

$$0 = \delta S = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)\, dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

Definition 9 (Lagrangian, Action)**.** *The function L is called the Lagrangian of the system concerned, and the integral S is called the action.*

We can derive the equations of motion from the principle of stationary action.

Theorem 10 (Lagrange's Equations)**.** *Consider a variation $q \to q + \delta q$ with endpoints fixed $\delta q(t_1) = 0, \delta q(t_2) = 0$. By the principle of stationary action,*

$$0 = \delta S = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q})\, dt = [\frac{\partial S}{\partial \dot{q}}\delta q]_{t_1}^{t_2} + \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\delta q\, dt = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\delta q\, dt.$$

Since the variation is arbitrary,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

Example 11 (Lagrangian of a Particle)**.** *When a single particle moves in an external field, the general form of Lagrangian is*

$$L = \frac{1}{2}mv^2 - U(r, t)$$

and the equation of motion is

$$ma = -\frac{\partial U}{\partial r} = F,$$

recovering Newton's Second Law of Motion.

Definition 12 (Legendre Transformation)**.** *Suppose f is a function of x_i and define $p_i = \frac{\partial f}{\partial x_i}$. Then we have $\mathrm{d}f = p_i \mathrm{d}x_i$.*

For example, the Legendre's transform on variable x_1 is defined as $f^* := x_1 p_1 - f$ which satisfies

$$\mathrm{d}f^* = \mathrm{d}(x_1 p_1 - f) = x_1 \mathrm{d}p_1 - (p_2 \mathrm{d}x_2 + \cdots + p_n \mathrm{d}x_n).$$

If we do a Legendre's transform of the Lagrangian L on coordinates $\dot{q}$, we get the Hamiltonian.

Definition 13 (Hamiltonian, linear momentum, Hamilton's equations)**.** *The Hamiltonian H (which represents the total energy) is defined in terms of the Lagrangian L as follows:*

$$H = \sum_i p_i \dot{q}_i - L$$

where p_i is the generalized linear momentum conjugate to the generalized coordinate q_i, and is given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

The Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Definition 14 (Poisson Brackets)**.** *In canonical coordinates, (q_i, p_i) on the phase space, given two functions $f(p_i, q_i, t)$ and $g(p_i, q_i, t)$, the Poisson bracket takes the form:*

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

The Poisson brackets of the canonical coordinates are:

$$\{q_k, q_l\} = 0, \quad \{p_k, p_l\} = 0, \quad \{q_k, p_l\} = \delta_{kl}.$$

where δ_{ij} is the Kronecker delta.

Remark 15 (Hamilton's equations of motion)**.** *Let $f(p, q, t)$ be some function of coordinates, momenta and time. By Hamilton's equations,*

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\},$$

so for a conserved quantity not explicitly dependent on time $\frac{df}{dt} = \{f, H\}$.

Theorem 16 (Hamilton-Jacobi equation)**.** *Since $\delta S = [\frac{\partial S}{\partial \dot{q}}\delta q]_{t_1}^{t_2} = p\delta q$ on a stationary path, we have $\frac{\partial S}{\partial \dot{q}} = p$. From the principle of stationary action, we're finding the path that makes the action stationary. Once we've found this optimal path, the action S becomes a function of just the endpoint coordinates and times, so $\frac{\partial S}{\partial \dot{q}} = 0$. Therefore we have*

$$L = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\dot{q} = \frac{\partial S}{\partial t} + p\dot{q}.$$

Rearranging,

$$\frac{\partial S}{\partial t} = -H,$$

which is the Hamilton-Jacobi equation.

Remark 17 (Symplectic Geometry)**.** *Hamiltonian mechanics can be formulated using the language of symplectic geometry. Here we outline the key concepts and equations.*

Definition 18. *A **symplectic manifold** is a smooth manifold M equipped with a closed, nondegenerate 2-form ω, called the **symplectic form**. Specifically:*

- *M is a $2n$-dimensional smooth manifold.*
- *ω is a closed 2-form, i.e., $d\omega = 0$, where d denotes the exterior derivative.*
- *ω is nondegenerate, meaning that at every point $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is nondegenerate.*

In the standard formulation of Hamiltonian mechanics, the phase space is $\mathbb{R}^{2n}$ equipped with canonical coordinates (q^i, p_i) for $i = 1, \ldots, n$, where q^i are the generalized coordinates, and p_i are the conjugate momenta. The symplectic form in these coordinates is:

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

The Hamiltonian vector field X_H can be written in these coordinates as:

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \right).$$

X_H satisfies the following equation:

$$\omega(X_H, \cdot) = dH.$$

This means that for any vector field Y on M,

$$\omega(X_H, Y) = dH(Y).$$

The flow generated by the Hamiltonian vector field X_H describes the time evolution of the system. If $\gamma(t)$ is an integral curve of X_H (i.e., $\frac{d\gamma}{dt} = X_H$), then the system evolves according to this curve in the phase space.

Example 19 (Simple Harmonic Oscillator)**.** *Consider a simple harmonic oscillator with the Hamiltonian:*

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2,$$

where m is the mass and k is the spring constant. The canonical coordinates are q for position and p for momentum. The symplectic form is $\omega = dq \wedge dp$, and the Hamiltonian vector field X_H is:

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} = \frac{p}{m}\frac{\partial}{\partial q} - kq\frac{\partial}{\partial p}.$$

Hamilton's equations for the simple harmonic oscillator are:

$$\frac{dq}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = -kq.$$

3.2 Symmetry

Theorem 20 (Noether's theorem)**.** *Every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.*

Proof. The action $S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$.

For a continuous symmetry, the action remains invariant $\delta S = 0$ under the transformations $q \to q + \delta q$ and $t \to t + \delta t$.

When we transform coordinates, the action becomes:

$S' = \int_{t_1+\delta t_1}^{t_2+\delta t_2} L(q + \delta q, \frac{d(q+\delta q)}{d(t+\delta t)}, t + \delta t)d(t + \delta t)$

The change in action is:

$\delta S = S' - S$

First, we need to transform the integrand:

1. $L(q + \delta q, \frac{d(q+\delta q)}{d(t+\delta t)}, t + \delta t)$
2. $d(t + \delta t) = dt + d(\delta t) = dt$ (since δt is a constant infinitesimal)

For the velocity term: $\frac{d(q+\delta q)}{d(t+\delta t)} \approx \frac{dq+d(\delta q)}{dt} \approx \dot{q} + \frac{d(\delta q)}{dt} - \dot{q}\frac{d(\delta t)}{dt} = \dot{q} + \frac{d}{dt}(\delta q) - \dot{q}\frac{d}{dt}(\delta t)$

Expanding L to first order: $L(q+\delta q, \dot{q}+\frac{d}{dt}(\delta q)-\dot{q}\frac{d}{dt}(\delta t), t+\delta t) \approx L+\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot{q}}(\frac{d}{dt}(\delta q)-\dot{q}\frac{d}{dt}(\delta t)) + \frac{\partial L}{\partial t}\delta t$

The change in action becomes: $\delta S = \int_{t_1}^{t_2}[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}(\frac{d}{dt}(\delta q) - \dot{q}\frac{d}{dt}(\delta t)) + \frac{\partial L}{\partial t}\delta t]dt + L(t_2)\delta t_2 - L(t_1)\delta t_1$

From the Euler-Lagrange equation, we know: $\frac{\partial L}{\partial q} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})$

Using integration by parts for the term with $\frac{d}{dt}(\delta q)$: $\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q)dt = [\frac{\partial L}{\partial \dot{q}}\delta q]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})\delta q dt$

Substituting and simplifying: $\delta S = \int_{t_1}^{t_2}[\frac{\partial L}{\partial q}\delta q - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})\delta q - \frac{\partial L}{\partial \dot{q}}\dot{q}\frac{d}{dt}(\delta t) + \frac{\partial L}{\partial t}\delta t]dt + [\frac{\partial L}{\partial \dot{q}}\delta q]_{t_1}^{t_2} + L(t_2)\delta t_2 - L(t_1)\delta t_1$

The first term cancels due to Euler-Lagrange equation, leaving: $\delta S = \int_{t_1}^{t_2}[-\frac{\partial L}{\partial \dot{q}}\dot{q}\frac{d}{dt}(\delta t)+\frac{\partial L}{\partial t}\delta t]dt + [\frac{\partial L}{\partial \dot{q}}\delta q]_{t_1}^{t_2} + L(t_2)\delta t_2 - L(t_1)\delta t_1$

If δt is constant, $\frac{d}{dt}(\delta t) = 0$, so this simplifies to: $\delta S = \int_{t_1}^{t_2} \frac{\partial L}{\partial t}\delta t dt + [\frac{\partial L}{\partial \dot{q}}\delta q]_{t_1}^{t_2} + L(t_2)\delta t_2 - L(t_1)\delta t_1$

For general symmetries where δt can vary with time, we integrate by parts: $\int_{t_1}^{t_2}[-\frac{\partial L}{\partial \dot{q}}\dot{q}\frac{d}{dt}(\delta t)]dt = [\frac{\partial L}{\partial \dot{q}}\dot{q}\delta t]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}\dot{q})\delta t dt$

Using the Hamiltonian definition: $H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$ and the momentum definition: $p = \frac{\partial L}{\partial \dot{q}}$

We get: $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}\dot{q}) = \frac{d}{dt}(p\dot{q}) = \frac{d}{dt}(H+L) = \frac{dH}{dt} + \frac{dL}{dt}$

For conservative systems: $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\partial L}{\partial t} + \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\partial L}{\partial t} + \frac{d}{dt}(p\dot{q}) - p\dot{q} = \frac{\partial L}{\partial t}$

So: $\frac{d}{dt}(p\dot{q}) = \frac{dH}{dt} + \frac{\partial L}{\partial t}$

Substituting back: $\delta S = -\int_{t_1}^{t_2}[\frac{dH}{dt} + \frac{\partial L}{\partial t} - \frac{\partial L}{\partial t}]\delta t dt + [p\delta q - p\dot{q}\delta t]_{t_1}^{t_2} + L(t_2)\delta t_2 - L(t_1)\delta t_1$

$\delta S = -\int_{t_1}^{t_2}\frac{dH}{dt}\delta t dt + [p\delta q - p\dot{q}\delta t + L\delta t]_{t_1}^{t_2}$

$\delta S = -\int_{t_1}^{t_2}\frac{dH}{dt}\delta t dt + [p\delta q - (p\dot{q} - L)\delta t]_{t_1}^{t_2}$

$\delta S = -\int_{t_1}^{t_2}\frac{dH}{dt}\delta t dt + [p\delta q - H\delta t]_{t_1}^{t_2}$

For symmetries where $\delta S = 0$: $\int_{t_1}^{t_2}\frac{dH}{dt}\delta t dt = [p\delta q - H\delta t]_{t_1}^{t_2}$

Assuming δt is constant, this implies: $\frac{d}{dt}(H\delta t - p\delta q) = 0$

Therefore, the quantity $H\delta t - p\delta q$ is conserved under the continuous symmetry transformation $q \to q + \delta q, t \to t + \delta t$. □

Definition 21 (Angular Momentum)**.** *The angular momentum* **J** *in terms of the Lagrangian L can be expressed as:*

$$\mathbf{J} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

where the generalized coordinates q_i are chosen to be $\mathbf{r}_i$, the position vector of the i-th particle in Cartesian coordinates, and $\mathbf{p}_i$ is its associated linear momentum given by:

$$\mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{r}}_i}$$

Example 22 (Conservation of Energy, Linear Momentum, and Angular Momentum)**.**

1. Conservation of Energy

Let's consider a symmetry where time is shifted uniformly: $t \to t+\delta t$ where δt is constant, but spatial coordinates remain unchanged ($\delta q = 0$).

The conserved quantity becomes: $H\delta t - p\delta q = H\delta t - 0 = H\delta t$

Since δt is a constant, this means H must be conserved. This is the conservation of energy, as H is the Hamiltonian of the system, which represents the total energy.

2. Conservation of Linear Momentum

Now consider a symmetry where space is uniformly shifted: $q \to q + \delta q$ where δq is constant, but time remains unchanged ($\delta t = 0$).

The conserved quantity becomes: $H\delta t - p\delta q = 0 - p\delta q = -p\delta q$

Since δq is a constant, this means p must be conserved. This is the conservation of linear momentum.

3. Conservation of Angular Momentum

Consider a symmetry where space undergoes an infinitesimal rotation. For rotational symmetry in a plane, we can write:

$q_1 \to q_1 + \delta\theta \cdot q_2$

$q_2 \to q_2 - \delta\theta \cdot q_1$

$\delta t = 0$

Where $\delta\theta$ is an infinitesimal rotation angle.

The conserved quantity becomes: $H\delta t - p_1\delta q_1 - p_2\delta q_2 = 0 - p_1(\delta\theta \cdot q_2) - p_2(-\delta\theta \cdot q_1)$ $= -\delta\theta(p_1q_2 - p_2q_1)$

Since $\delta\theta$ *is a constant, the quantity* $(p_1q_2 - p_2q_1)$ *must be conserved. This is the conservation of angular momentum in the plane of rotation, which can be written as* $L = r \times p$ *in vector form, where the generalized coordinates* q *are chosen to be the Cartesian coordinates* r*.*

The conserved quantity $H\delta t - p\delta q$ *is the general form for these conservation laws, where different symmetries lead to different conserved quantities by specifying particular forms of* δt *and* δq*.*

Part II

Classical Field Theory

Chapter 4

Electromagnetism

4.1 Early Developments

- **600 BCE**: Thales of Miletus observed that amber, when rubbed with cloth, could attract light objects (static electricity).
- **230 BCE**: Greek shepherds noticed that certain stones (lodestones) could attract iron (magnetism).
- **1600**: William Gilbert published *De Magnete*, differentiating between magnetism and static electricity and suggesting that the Earth itself is a giant magnet.
- **1752**: Benjamin Franklin conducted his famous kite experiment to study lightning, establishing that lightning is a form of electricity.
- **1820**: Hans Christian Ørsted discovered that an electric current deflects a magnetic compass needle, demonstrating a direct relationship between electricity and magnetism.
- **1821**: André-Marie Ampère formulated Ampère's Law, describing the magnetic force between two electric currents.
- **1831**: Michael Faraday discovered electromagnetic induction, showing that a changing magnetic field can induce an electric current.
- **1864**: James Clerk Maxwell formulated Maxwell's equations, which mathematically unified electricity, magnetism, and light as manifestations of the same fundamental force: electromagnetism.
- **1888**: Heinrich Hertz experimentally confirmed the existence of electromagnetic waves, as predicted by Maxwell.

4.2 Electromagnetic Fields

Within the framework of classical theory elementary particles must be treated as points. The interaction of particles can be described with the help of the concept of a field of force. Namely, instead of saying that one particle acts on another, we may say that the particle creates a field around itself; a certain force then acts on every other particle

located in this field. The forces acting on a particle at a given moment are not determined by the positions at that same moment. A change in the position of one of the particles influences other particles only after the lapse of a certain time interval. This means that the field itself acquires physical reality. We cannot speak of a direct interaction of particles located at a distance from one another. Interactions can occur at any one moment only between neighboring points in space (contact interaction). Therefore we must speak of the interaction of the one particle with the field, and of the subsequent interaction of the field with the second particle.

In this chapter we consider electromagnetic fields.

Remark 23 (action of the whole electromagnetic system)**.** *The action function S for the whole system, consisting of an electromagnetic field as well as the particles located in it, must consist of three parts:*

$$S = S_m + S_{mf} + S_f,$$

where $S_m = -mc\int ds$ is that part of the action which depends only on the properties of the particles, that is, just the action for free particles;

S_{mf} is that part of the action which depends on the interaction between the particles and the field;

and S_f is the action for the field.

Definition 24 (Lagrangian for a Charged Particle, four-potential)**.** *For a particle moving in a given electromagnetic field, the action is made up of two parts: the action for the free particle, and a term describing the interaction of the particle with the field. Classically, based on experimental data, we observe that the properties of a particle with respect to interaction with the electromagnetic field are determined by a single parameter—the charge e of the particle, which can be either positive or negative (or equal to zero).*

The properties of the field are characterized by a four-vector $A_\mu = (\phi, A)$, the four-potential, whose components are functions of the coordinates and time. Here ϕ is called the scalar potential and A is called the vector potential of the field.

So the interaction term $S_{mf} = \int -\frac{e}{c}A_\mu dx^\mu$.

Thus the action function for a charge in an electromagnetic field has the form

$$S = S_m + S_{mf} = \int -mc\,ds - \frac{e}{c}A_\mu dx^\mu.$$

The Lagrangian is

$$L = -mc^2\gamma^{-1} + \frac{e}{c}A\cdot v - e\phi.$$

The Hamiltonian is

$$H = v\cdot\frac{\partial L}{\partial v} - L = mc^2\gamma^{-1} + e\phi.$$

Definition 25 (electromagnetic field tensor, electric field, magnetic field)**.** *The covariant form of the electromagnetic field tensor $F_{\mu\nu}$ is defined as:*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and the contravariant form is related by

$$F^{\mu\nu} = \eta^{\alpha\nu}F_{\mu\nu}\eta^{\beta\mu}.$$

The components of F relate to the electric fields E and magnetic fields B as follows:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

The Hodge dual of $F_{\mu\nu}$ is

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma},$$

Remark 26 (Lorentz invariants of the electromagnetic field)**.** *The scalar invariant is:*

$$F_{\mu\nu}F^{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right),$$

and the pseudoscalar invariant is:

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}.$$

Instead of treating charges as points, for mathematical convenience we frequently consider them to be distributed continuously in space. Then we can introduce the "charge density" ρ such that ρdV is the charge contained in the volume dV.

Definition 27 (four-current)**.**

$$J^\mu = (\rho c, \mathbf{J} = \rho v) = \rho\frac{dx^\mu}{dt}$$

To establish the form of the action S_f for the field, we start from the following very important property of electromagnetic fields.

Theorem 28 (Principle of Superposition)**.** *As experiment shows, the field produced by a system of charges is the result of a simple composition of the fields produced by each of the particles individually. This means that the resultant field intensity at each point is equal to the vector sum of the individual field intensities at that point.*

As is well known, linear differential equations have just this property, that the sum of any solutions is also a solution. Consequently the field equations shall be linear differential equations. It follows that under the integral sign for the action S_f there must stand an expression quadratic in the field. Only in this case will the field equations be linear; the field equations are obtained by varying the action, and in the variation the degree of the expression under the integral sign decreases by unity. The potentials A_μ cannot enter into the expression for the action S_f , since they are gauge invariant and thus not uniquely determined. Therefore S_f must be the integral of some function of the electromagnetic field tensor F. But the action must be a scalar and must therefore be the integral of some scalar that satisfies Lorentz invariance. The only such quantity is the product $F_{\mu\nu}F^{\mu\nu}$.

Definition 29 (Lagrangian for the Electromagnetic Field)**.** *Write $dxdydz = dV$, $cdtdV = d\Omega$,*

$$L_f = -\frac{1}{16\pi}\int F_{\mu\nu}F^{\mu\nu}dV = \frac{1}{8\pi}\int\left(E^2 - B^2\right)dV$$

The action for the field is:

$$S_f = \int Ldt = -\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\,d\Omega.$$

The action for the whole system is now

$$S = -\int mc\,ds - \int \frac{e}{c} A_\mu\,dx^\mu - \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\,d\Omega$$

if you treat charges as particles, or

$$S = -\int mc\,ds - \int \frac{1}{c^2} A_\mu j^\mu\,d\Omega - \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\,d\Omega$$

if you replace point charges as a continuous distribution of charge with density.

4.3 Gauge Invariance

Electromagnetism exhibits gauge invariance under the local U(1) symmetry group. This means the physical observations remain unchanged under the following gauge transformations of the four-potential

$$A_\mu \to A_\mu + \partial_\mu \lambda$$

where $\lambda(x)$ is an arbitrary scalar function of spacetime coordinates.

Definition 30 (Unitary Group of Degree 1 U(1))**.**

$$U(1) = \{z = e^{i\theta} \in \mathbb{C} \mid |z| = 1\}.$$

The $U(1)$ group is a continuous (Lie) group,meaning it has a smooth manifold struc- ture, and it is abelian.

Remark 31 (U(1) symmetry and gauge transformations)**.** *We demonstrate that the transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ is precisely what's needed to maintain U(1) gauge invariance of a theory with charged fields. It's the complement to the direct U(1) transformation on the matter fields.*

While the matter fields transform directly under U(1), the gauge field A_μ transforms in a way that preserves the overall U(1) invariance of the complete system.

1. *Matter Field Transformation: When we apply a local U(1) transformation, a charged field transforms as: $\psi(x) \to \psi'(x) = e^{iq\lambda(x)}\psi(x)$*

   This is directly a U(1) transformation where the group element $e^{iq\lambda(x)}$ acts on ψ.

2. *The standard derivative has an extra term that breaks the symmetry in the Lagrangian. The covariant derivative $D_\mu = \partial_\mu - iqA_\mu$ allows ψ to transform as: $D_\mu\psi \to D'_\mu\psi' = e^{iq\lambda}D_\mu\psi$ when the gauge field transforms as: $A_\mu \to A'_\mu = A_\mu + \partial_\mu\lambda$.*

3. *Explicitly, $D'_\mu\psi' = (\partial_\mu - iqA'_\mu)(e^{iq\lambda}\psi) = \partial_\mu(e^{iq\lambda}\psi) - iq(A_\mu + \partial_\mu\lambda)(e^{iq\lambda}\psi) = e^{iq\lambda}\partial_\mu\psi + iq(\partial_\mu\lambda)e^{iq\lambda}\psi - iqA_\mu e^{iq\lambda}\psi - iq(\partial_\mu\lambda)e^{iq\lambda}\psi = e^{iq\lambda}(\partial_\mu\psi - iqA_\mu\psi) = e^{iq\lambda}D_\mu\psi$*

Theorem 32 (conservation of charge, equation of continuity)**.** *The gauge invariance of the Lagrangian of electromagnetism implies the conservation of electric charge.*

Note that $F_{\mu\nu}$ is unchanged under gauge transformation, so the only term left is:

$$0 = \delta S = -\frac{1}{c^2}\int \partial_\mu\lambda j^\mu\,d\Omega.$$

Since λ is arbitrary, $\partial_\mu \lambda j^\mu = 0$. Taking its difference with a four-divergence $\partial_\mu(\lambda j^\mu)$ (using Gauss' theorem, the integral of it over the four-volume is transformed into an integral over the bounding hypersurface ; on varying the action, these integrals drop out and thus have no effect on the equations of motion), we get the equation of continuity:

$$\partial_\mu j^\mu = \frac{\partial j^\mu}{\partial x^\mu} = 0.$$

4.4 Electrodynamics

We shall find the equations of motion of a charge in a given electromagnetic field, neglecting the action of the charge on the field can be neglected, assuming the charge e is not large.

Theorem 33 (Lorentz force)**.** *Given the Lagrangian of a charged particle, the Lagrange equation gives us*

$$mc\frac{du_\mu}{ds} = \frac{e}{c}F^{\mu\nu}u_\nu,$$

which can also be written as

$$\frac{dp}{dt} = eE + \frac{e}{c}v \times B,$$

where the expression on the right is called the Lorentz force.

To find the equations of motion we assumed the field to be given and varied the trajectory of the particle. On the other hand, in finding the field equations with the aid of the principle of least action we must assume the motion of the charges to be given and vary only the potentials (which serve as the "coordinates" of the system).

Theorem 34 (Maxwell equations)**.** *The Maxwell equations in covariant form are:*

1. ***Gauss's Law for Electricity and Ampère's Law (with Maxwell's addition) combined****: The inhomogeneous pair of Maxwell's equations is derived from the variation of the Lagrangian density with respect to A_μ:*

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c}J^\mu$$

(we are using Gaussian units, if using SI units replace $\frac{4\pi}{c}$ by μ_0, the permeability of free space).

This pair corresponds to

$$\nabla \cdot \mathbf{E} = 4\pi\rho,$$
$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}.$$

2. ***Faraday's Law of Induction and Gauss's Law for Magnetism combined****:*

The homogeneous Maxwell's equations is a result of the antisymmetry of the field tensor $F_{\mu\nu}$:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0,$$

which corresponds to

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0.$$

Example 35 (Dipole and Multipole Moments)**.** *We consider the field produced by a system of charges at large distances, that is, at distances large compared with the dimensions of the system. We introduce a coordinate system with the origin anywhere within the system of charges. Let the radius vectors of the various charges be* $\mathbf{r}_a$*. The potential of the field produced by all the charges at the point having the radius vector* $\mathbf{R}$ *is*

$$\Phi(\mathbf{R}) = \sum_a \frac{e_a}{|\mathbf{R} - \mathbf{r}_a|},$$

where $|\mathbf{R} - \mathbf{r}_a|$ *are the radius vectors from the charges* e_a *to the point where we are finding the potential.*

We must investigate this expansion for large $\mathbf{R}$ *(*$\mathbf{R} >> \mathbf{r}_a$*). To do this, we expand it in powers of* $\frac{\mathbf{r}_a}{\mathbf{R}}$*, to terms of first order,*

$$f(\mathbf{R} - \mathbf{r}) = f(\mathbf{R}) - \mathbf{r} \cdot \nabla f(\mathbf{R}),$$

where, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

$$\Phi \approx \frac{1}{R} \sum_a e_a - \sum_a e_a \mathbf{r}_a \cdot \nabla \frac{1}{R}.$$

The sum

$$\mathbf{d} = \sum_a e_a \mathbf{r}_a$$

is called the dipole moment of the system of charges.

It is important to note that if the sum of all the charges, $\sum_a e_a$*, is zero, then the dipole moment does not depend on the choice of the origin of coordinates.*

In the expansion of the potential in powers of $\frac{1}{R}$*,*

$$\Phi = \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \ldots,$$

the term $\Phi^{(n)}$ *is proportional to* $\frac{1}{R^{n+1}}$*. We saw that the first term,* $\Phi^{(0)}$*, is determined by the sum of all the charges; the second term,* $\Phi^{(1)}$*, is determined by the dipole moment of the system.* $\Phi^{(2)}$ *is usually called the quadrupole potential.*

4.5 Energy-Momentum Tensor

4.5.1 Global Symmetry of the Field

Recall from Lagrangian mechanics that time-translational invariance leads to energy conservation and spatial translation symmetry results in momentum conservation. In classical field theory, the energy-momentum tensor is conserved due to spacetime translation symmetry:

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu,$$

where ϵ^μ is a constant 4-vector. Under this transformation, the Lagrangian density transforms as follows:

$$\mathcal{L}(x) = \mathcal{L}(x' + \epsilon) = \mathcal{L}(x') + \epsilon^\mu \partial_\mu \mathcal{L}.$$

Definition 36 (energy-momentum tensor)**.** *Setting $\delta S = 0$, we find the conserved quantity, which we call the energy-momentum tensor, is*

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)} \partial^\nu \phi_\alpha - \delta^{\mu\nu} \mathcal{L},$$

where ϕ^α represents the fields.

The equation for the conservation of the energy-momentum tensor is given by:

$$\partial_\mu T^{\mu\nu} = 0.$$

4.5.2 electromagnetic energy-momentum tensor

In electromagnetism, the energy-momentum tensor of the whole system is the sum of the energy-momentum tensors for the electromagnetic field and for the particles, where in the latter the particles are assumed not to interact with one another.

Definition 37 (Energy-Momentum tensor of the electromagnetic field in the absence of charges, energy density, Poynting vector)**.** *The Lagrangian density for the electromagnetic field is: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, and the field $\phi_\mu = A_\mu$.*

Adding a surface term for simplicity, the energy-momentum tensor for the electromagnetic field is given by:

$$T^{\mu\nu} = \frac{1}{4\pi}(-F^{\mu\alpha} F^{\nu}{}_{\alpha} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}).$$

Explicitly in matrix form, the stress-energy tensor $T^{\mu\nu}$ is given by:

$$T^{\mu\nu} = \begin{bmatrix} u & \frac{1}{c} S_x & \frac{1}{c} S_y & \frac{1}{c} S_z \\ \frac{1}{c} S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{1}{c} S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{1}{c} S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix},$$

where

$$u = \frac{1}{8\pi} \left(\mathbf{E}^2 + \mathbf{B}^2 \right)$$

is the energy density of the electromagnetic field,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

is the Poynting vector, where the Poynting vector S describes the energy flux (energy per unit area per unit time), and σ_{ij} is the Maxwell stress tensor.

Definition 38 (Energy-Momentum tensor of the system of non-interacting particles)**.**

$$T^{\mu\nu} = \mu c \frac{dx^\mu}{ds} \frac{dx^\nu}{dt},$$

where the mass density in the form $\mu = \sum m_{particle} \delta(r - r_{particle})$.

4.5.3 energy-momentum tensor for a perfect fluid

In addition to the energy-momentum tensor for a system of point particles, we shall also need the expression for this tensor for macroscopic bodies which are treated as being continuous.

Now we introduce a reference system in which a given element of volume of the body is at rest. In such a reference system, Pascal's law is valid, that is, the pressure p applied to a given portion of the body is transmitted equally in all directions and is everywhere perpendicular to the surface on which it acts. So the stress tensor is $p\delta_{\alpha\beta}$.

The component T_{00} of the energy-momentum tensor $T_{\mu\nu}$ represents the energy density e of the body. The quantity $\frac{e}{c^2}$ is then the mass density ρ, which is the mass per unit volume. $T_{\alpha 0}$ the momentum density, they are equal to zero in this reference frame.

Thus, in the reference system under consideration, the energy-momentum tensor $T_{\mu\nu}$ for the given portion of the body has the form:

$$T_{\mu\nu} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

Now it is easy to find the expression for the energy-momentum tensor in an arbitrary reference system. To do this we introduce the four-velocity u^μ for the macroscopic motion of an element of volume of the body. In the rest frame of the particular element, $u^\mu = (1, 0)$. From this we get the general form of the energy-momentum tensor for a perfect fluid:

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu}.$$

4.6 Electromagnetic Waves

4.6.1 Waves

Definition 39 (electromagnetic waves)**.** *Electromagnetic waves are generated whenever charged particles accelerate. Even though charges are the source of electromagnetic waves (through acceleration), once the wave is created, it can propagate independently of the source. This is because the electric and magnetic fields are self-sustaining in a way that changing electric fields produce magnetic fields, and vice versa, creating a self-propagating wave that moves through space. This mutual interaction allows EM waves to propagate without needing any ongoing charge acceleration in the region where the wave exists.*

So we study the electromagnetic field in vacuum and let $\rho = 0, j = 0$ in Maxwell's equations. The electromagnetic fields occurring in vacuum in the absence of charges are called the electromagnetic waves.

Theorem 40 (the wave equation)**.** *Choose the temporal gauge $A^0 = \phi = 0$ and Lorentz gauge $\partial_\mu A^\mu = 0$, from the Maxwell's equations we get the wave equation*

$$\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}\right) - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0.$$

Definition 41 (plane waves, transverse)**.** *If the field of electromagnetic waves only depends on one coordinate x and on time t, it is called plane waves.*

The wave equation becomes

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})f = 0.$$

To solve this equation, we introduce new variables

$$\xi = t - \frac{x}{c}, \quad \eta = t + \frac{x}{c}$$

so that the equation becomes

$$\frac{\partial^2 f}{\partial \xi \partial \eta} = 0.$$

The solution obviously has the form

$$f = f_1(\xi) + f_2(\eta),$$

where f_1 and f_2 are arbitrary functions. Thus,

$$f = f_1\left(t - \frac{x}{c}\right) + f_2\left(t + \frac{x}{c}\right).$$

It is clear that the field has the same values for coordinates x and times t which satisfy the relation $t - (x/c) = const$. This means that if, at some time $t = 0$, the field at a certain point x in space had some definite value, then after an interval of time t the field has that same value at a distance ct along the X axis from the original place. We can say that all the values of the electromagnetic field are propagated in space along the X axis with a velocity equal to the velocity of light, c. Thus, $f_1\left(t - \frac{x}{c}\right)$ represents a plane wave moving in the positive direction along the X axis, and $f_2\left(t + \frac{x}{c}\right)$ represents a wave moving in the negative direction along the X axis.

Since the Lorentz gauge gives us $\frac{\partial A_x}{\partial x} = 0$ and the wave equation gives us $\frac{\partial A_x}{\partial t}$ =const, we can set $A_x = 0$. Thus the vector potential of the plane wave can always be chosen perpendicular to the direction of propagation of that wave.

From the equations we get

$$B = n \times E,$$

where n is a unit vector along the direction of propagation of the wave.

Therefore, we say electromagnetic waves are transverse, i.e. the electric and magnetic fields E and B of a plane wave are directed perpendicular to the direction of propagation of the wave.

Definition 42 (monochromatic plane waves, frequency, wavelength, wave vector, phase)**.** *If the field is a periodic function of the time, such a wave is said to be monochromatic.*

The vector potential A of such a monochromatic plane wave can be written as the real part of a complex expression

$$A = Re(A_0 e^{-i\omega(t-\frac{x}{c})}) = Re(A_0 e^{i(kr-\omega t)}),$$

where along x-axis $r = (x, 0, 0)$.

- *frequency: ω*
- *wave length: $\lambda = \frac{2\pi c}{\omega}$*

- *wave vector:*
$$k^\mu = (\frac{\omega}{c}, \mathbf{k}), \quad \mathbf{k} = \frac{w}{c}\mathbf{n},$$
where n is a unit vector along the direction of propagation of the wave. For $x^\mu = (t, r), \quad k_\mu x^\mu = \omega t - \mathbf{k}r$. *In particular,* $k_\mu k^\mu = 0$.

- *phase:* $\mathbf{k} \cdot r - \omega t$

4.6.2 Spectrum and Fourier Analysis

Every wave can be represented as a superposition of monochromatic waves with various frequencies. The Fourier expansion gives

$$f(t) = \int \frac{1}{2\pi} f_\omega e^{-i\omega t}\, d\omega, \quad f_\omega = \int f(t) e^{i\omega t}\, dt.$$

Note that $f(t)$ must be real, so $f_{-\omega} = f_\omega^*$, here $*$ denotes the complex conjugate. The average intensity of the wave is

$$\int_{-\infty}^{\infty} f^2\, dt = 2\int_0^\infty \frac{1}{2\pi}|f_\omega|^2\, d\omega.$$

Example 43 (Polarization)**.** *We now treat in more detail the direction of the field of a monochromatic wave. To be specific, we shall talk of the electric field* $\mathbf{E} = Re\{\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\}$ *(everything stated below applies equally well, of course, to the magnetic field). Define*

$$\mathbf{E}_0 = \mathbf{b}e^{-i\alpha}, \quad \mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2, \quad \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}^2 \in \mathcal{R}.$$

With this definition, we write:

$$\mathbf{E} = Re\{\mathbf{b}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t-\alpha)}\}.$$

Since $\mathbf{b}^2$ *is chosen to be real, it follows that* $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$, *i.e. the vectors* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *are mutually perpendicular. We choose the direction of* $\mathbf{b}_1$ *as the y axis (and the x axis along the direction of propagation of the wave). We then have from:*

$$E_y = b_1 \cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \alpha), \qquad E_z = \pm b_2 \sin(\omega t - \mathbf{k}\cdot\mathbf{r} + \alpha),$$

where we use the plus (minus) sign if $\mathbf{b}_2$ *is along the positive (negative) z axis.*

Therefore, every monochromatic wave is polarized.

- *If* $b_1 = 0$ *or* $b_2 = 0$, *the wave is said to be linearly polarized, or plane polarized.*

- *Else we can write*
$$\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1.$$
Thus we see that, at each point in space, the electric field vector rotates in a plane perpendicular to the direction of propagation of the wave, while its endpoint describes the ellipse.

- *If* $b_1 = b_2$, *the wave is circularly polarized.*

- *If not, such a wave is said to be elliptically polarized.*

  An elliptically polarized wave can be treated as the superposition of two plane polarized waves.

we usually have to deal with waves which are only approximately monochromatic, and let ω be some average frequency for it. Then its field (to be specific we shall consider the electric field $\mathbf{E}$) at a fixed point in space can be written in the form

$$\mathbf{E}(t) = Re\{\mathbf{E}_0(t)e^{i\omega t}\},$$

where the complex amplitude $\mathbf{E}_0(t)$ is some slowly varying function of time (for a strictly monochromatic wave $\mathbf{E}_0$ would be constant). Since $\mathbf{E}_0$ determines the polarization of the wave, this means that at each point of the wave, its polarization changes with time; such a wave is said to be partially polarized.

The polarization properties of electromagnetic waves, and of light in particular, are observed experimentally by passing the light to be investigated through various bodies and then observing the intensity of the transmitted light. From the mathematical point of view this means that we draw conclusions concerning the polarization properties of the light from the values of certain quadratic functions of its field. Here, of course, we are considering the time averages of such functions. Thus we see that the polarization properties of the light are completely characterized by the tensor

$$J_{\alpha\beta} = \langle E_\alpha E^*_\beta \rangle.$$

Since the vector $\mathbf{E}$ always lies in a plane perpendicular to the direction of the wave, the tensor $J_{\alpha\beta}$ has altogether four components (the indices α, β take on only two values corresponding to the y and z axes; the x axis is along the direction of propagation of the wave).

The sum of the diagonal elements of the tensor $J_{\alpha\beta}$ (we denote it by J) is a real quantity:

$$J = \sum_\alpha J_{\alpha\alpha} = \langle \mathbf{E}^2 \rangle.$$

This quantity determines the intensity of the wave, as measured by the energy flux density.

To eliminate this quantity which is not directly related to the polarization properties, we introduce in place of $J_{\alpha\beta}$ the polarization tensor

$$\rho_{\alpha\beta} = \frac{J_{\alpha\beta}}{J}.$$

- *For completely polarized light, $E_0 =$const,* $\det \rho_{\alpha\beta} = 0$.

- *For unpolarized or natural light, complete absence of polarization means that all directions in the y-z plane are equivalent.* $\rho_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}, \quad \det \rho_{\alpha\beta} = \frac{1}{4}$.

The degree of polarization P is the positive quantity such that $\frac{1}{4}(1 - P^2) = \det \rho_{\alpha\beta}$.

4.6.3 Field of Moving Charges

Electromagnetic waves are the only non-trivial solutions of Maxwell equations in vacuum, without sources. Furthermore, with static sources (charges at rest and steady-state currents) electric and magnetic fields are also constant in time. Now we study time-dependent solutions to Maxwell's equations in the presence of time-varying sources.

Theorem 44 (retarded potential, Liénard–Wiechert potential)**.** *The potential we observe at a given position r and time t comes from sources moving at earlier times $t' = t - R/c$, since it takes a finite amount of time for light to propagate and convey the information of the motion of the charges. Because of this effect of retardation in time, the solution is known as a retarded potential.*

For the potentials of the field produced by an arbitrarily moving point charge e, the following expressions, called the Lienard-Wiechert potentials, are:

$$\phi = \frac{e}{(R - \frac{v \cdot \mathbf{R}}{c})}, \quad \mathbf{A} = \frac{ev}{c(R - \frac{v \cdot \mathbf{R}}{c})}, \tag{4.1}$$

where $\mathbf{R} = r - r^{charge}|_{t'}$, is taken from the point where the charge is located to the point of observation at retarded time t', $R = |\mathbf{R}|$, v is the velocity of the charge.

Using the Lienard-Wiechert potentials and the principle of superposition, we obtain the exact solution of the Maxwell equations for a given distribution of charges and currents.

4.6.4 Radiation

Radiation is the process by which energy is emitted and propagates through space in the form of waves or particles.

Radiation of electromagnetic waves is the emission of self-propagating electric and magnetic fields, usually caused by accelerating charges.

Let the unit vector from the system of charges to the observation point is n, and the distance between the observation point and the system of charges is R.

At sufficiently large distances from the system of charges, the field over small regions of space can be considered to be a plane wave.

$$B = \dot{A} \times n, \quad E = B \times n.$$

The radiated electromagnetic waves carry off energy. The density dI of radiation into the solid angle $d\Omega$ is $dI = SR^2\, d\Omega = \frac{cB^2}{4\pi} R^2\, d\Omega$, where S is the magnitude of the Poynting vector $\mathbf{S}$, i.e. the energy flux density.

Example 45 (Dipole Radiation)**.** *Now we study radiation at large distances from the system of radiating charges, much larger than the wavelength of the radiated electromagnetic waves. We shall assume also that the wavelength is much larger than the physical size of the system of charges.*

Now the retarded potential has the form

$$A = \frac{1}{cR} \int \mathbf{j}|'_t \, dV = \frac{1}{cR} \sum ev = \frac{1}{cR} \dot{d},$$

where d is the dipole moment. We find the magnetic and electric fields are

$$B = \frac{1}{c^2 R} \ddot{d} \times n, \quad E = \frac{1}{c^2 R} \ddot{d} \times n \times n.$$

Radiation of this kind is called dipole radiation.

Since $\ddot{d} = \sum ea$, charges can only radiate if they accelerate. This also follows directly from the principle of relativity, since a charge in uniform motion can be considered in the inertial system in which it is at rest, and a charge at rest does not radiate.

The intensity of dipole radiation is

$$dI = \frac{1}{4\pi c^3} (\ddot{d} \times n)^2 d\Omega \; = \frac{\ddot{d}}{4\pi c^3} \sin^2 \theta d\Omega.$$

Example 46 (Scattering)**.** *If an electromagnetic wave falls on a system of charges, then under its action the charges are set in motion. This motion in turn produces radiation in all directions; there occurs, we say, a scattering of the original wave.*

The scattering is most conveniently characterized by the ratio of the amount of energy emitted by the scattering system in a given direction per unit time, to the energy flux density of the incident radiation. This ratio clearly has dimensions of area, and is called the effective scattering cross-section (or simply the cross-section).

Let dI be the energy radiated by the system into solid angle $d\Omega$ per second, and let dP be the total power radiated into solid angle $d\Omega$ for an incident wave with Poynting vector S. Then the effective cross-section for scattering (into the solid angle $d\Omega$) is

$$d\sigma = \frac{dI}{S} = \frac{dP}{S \cdot d\Omega}.$$

Let us consider the scattering produced by a free charge at rest. Suppose there is incident on this charge a plane monochromatic linearly polarized wave. Its electric field can be written in the form $E = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \alpha)$. We shall assume that the velocity acquired by the charge under the influence of the incident wave is small compared with the velocity of light (which is usually the case). Then we can consider the force acting on the charge to be eE, while the force $\left(\frac{e}{c}\mathbf{v} \times \mathbf{B}\right)$ due to the magnetic field can be neglected. In this case we can also neglect the effect of the displacement of the charge during its vibrations under the influence of the field. If the charge carries out vibrations around the coordinate origin, then we can assume that the field which acts on the charge at all times is the same as that at the origin, that is, $E = E_0 \cos(\omega t - \alpha)$. Since the equation of motion of the charge is

$$m\ddot{x} = eE$$

and its dipole moment $d = ex$, then

$$\ddot{d} = \frac{e^2}{m} E.$$

To calculate the scattered radiation, we use formula for dipole radiation. We also note that the frequency of the wave radiated by the charge (i.e., scattered by it) is clearly the same as the frequency of the incident wave. We find

$$dI = \left(\frac{dP}{d\Omega}\right)_{scattered} = \left(\frac{e^4}{4\pi m^2 c^3}\right) (E \times n)^2 \, d\Omega.$$

On the other hand, the Poynting vector of the incident wave is

$$S = \frac{c}{4\pi} E^2.$$

From this we find, for the cross-section for scattering into the solid angle $d\Omega$,

$$d\sigma = \left(\frac{e^2}{mc^2}\right)^2 \sin^2\theta \, d\Omega,$$

where θ is the angle between the direction of scattering (the vector $\mathbf{n}$), and the direction of the electric field $\mathbf{E}$ of the incident wave. We see that the effective scattering cross-section of a free charge is independent of frequency.

We determine the total cross-section σ. To do this, we choose the polar axis along $\mathbf{E}$. *Then* $d\Omega = \sin\theta\, d\theta\, d\phi$; *substituting this and integrating with respect to* θ *from* 0 *to* π, *and over* ϕ *from* 0 *to* 2π, *we find*

$$\sigma_T = \left(\frac{8\pi}{3}\right)\left(\frac{e^2}{mc^2}\right)^2.$$

(This is the Thomson formula.)

Chapter 5

Special Relativity

5.1 Early Developments

Remark 47 (Conflicts Between Electromagnetic Theory and Classical Mechanics)**.** *James Clerk Maxwell's equations of electromagnetism, formulated in the 1860s, showed that light is an electromagnetic wave traveling at a constant speed in a vacuum, regardless of the motion of the source or observer. This was in direct conflict with classical mechanics, based on Newtonian physics, which assumed that time and space were absolute and independent.*

Example 48 (Michelson-Morley Experiment (1887))**.** *This experiment, conducted by Albert A. Michelson and Edward W. Morley, aimed to detect the presence of the "aether," a supposed medium through which light waves traveled. The experiment found no evidence of the aether, indicating that the speed of light was invariant regardless of the motion of the observer. This result was inexplicable within the framework of classical mechanics.*

Remark 49 (Lorentz Transformations and the Work of Henri Poincaré)**.** *Hendrik Lorentz developed the Lorentz transformations in the late 19th century, with significant contributions occurring in the 1890s. Specifically, Lorentz formulated the transformations as part of his attempt to explain the results of the Michelson-Morley experiment and to reconcile them with Maxwell's equations.*

Henri Poincaré further developed these ideas in the early 1900s, particularly around 1904-1905. Poincaré recognized the importance of the Lorentz transformations and the need for a new conception of space and time, coming very close to formulating the principles underlying special relativity.

5.2 The Principle of Relativity

Theorem 50 (Postulates of special relativity)**.** *The whole theory is based on the following two postulates:*

- *Principle of Relativity: The laws of physics are the same in all inertial frames of reference. This means that no physical experiment can distinguish one inertial frame from another.*
- *Invariance of the Speed of Light: The speed of light $c \approx 3.00 \times 10^8$ m/s in a vacuum is constant and will be the same for all observers, regardless of the motion of the light source or the observer.*

Definition 51 (Four-vector, Minkowski metric tensor, spacetime interval, proper time, four-velocity)**.** *We give several definitions about spacetime.*

- *A four-vector is a vector in four-dimensional spacetime consisting of one time component and three spatial components. It can be written as:*

$$x^\mu = (ct, \vec{x}) = (ct, x, y, z)$$

where c is the speed of light, t is the time coordinate, and $\vec{x} = (x, y, z)$ are the spatial coordinates.

- *The Minkowski metric tensor $\eta_{\mu\nu}$ encapsulates the geometry of flat spacetime in special relativity.*

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- *the invariant spacetime interval s between two events:*

$$s^2 = \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) = (ct)^2 - x^2 - y^2 - z^2.$$

Also define ds to be the infinitesimal interval in Minkowski spacetime and is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

  - *Time-like interval: $ds^2 > 0$*
  - *Space-like interval: $ds^2 < 0$*
  - *Light-like (null) interval: $ds^2 = 0$*

- *Proper time, denoted by τ, is the time interval measured by a clock that is moving with the object in question. It is the time experienced by an observer who is at rest relative to the event being timed.*

  In special relativity, the proper time interval $d\tau$ between two events in an inertial frame can be related to measured in another inertial frame as follows:

$$d\tau = \frac{ds}{c} = \frac{dt}{\gamma},$$

  where the coordinate time interval is dt.

- *The four-velocity $u_\mu = \frac{dx_\mu}{d\tau}$.*

Definition 52 (Lorentz transformation)**.** *A Lorentz transformation is a linear transformation that relates the spacetime coordinates of events as observed in two different inertial frames of reference. Mathematically, a Lorentz transformation can be represented as:*

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

where x'^μ are the transformed coordinates, x^μ are the original coordinates, and $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix. This transformation preserves the spacetime interval:

$$x^\mu x_\mu = x'^\mu x'_\mu.$$

The Lorentz group consists of all linear transformations Λ that preserve the Minkowski metric tensor. This condition can be written mathematically as:

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta_{\mu\nu} = \eta_{\alpha\beta}.$$

Example 53 (boost)**.** *The Lorentz transformation matrix for a frame moving with velocity v along the x-axis relative to a stationary frame is:*

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}.$$

Alternatively, the theory of special relativity can be equivalently derived based on universal Lorentz Invariance.

Theorem 54 (Lorentz Invariance)**.** *The laws of physics are invariant with respect to Lorentz transformations.*

For example, consider a tensor equation: $T^{\mu\nu} = S^{\mu\nu}$.

For Lorentz invariance to hold, the transformed equation must satisfy

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}S^{\alpha\beta} = S'^{\mu\nu},$$

indicating that the form of the physical law does not change.

Additionally, if $\phi(x)$ represents a field quantity, Lorentz invariance requires that $\phi(x)$ transforms in such a way under a Lorentz transformation Λ that the form of the relevant equations are preserved:

$$\phi'(x') = \phi(\Lambda x).$$

Thus, Lorentz invariance can be concisely expressed with the condition that any physical law $\mathcal{L}$ under a Lorentz transformation satisfies:

$$\mathcal{L}(x) = \mathcal{L}(\Lambda x).$$

A law written in 4-vector or tensor notation is automatically Lorentz invariant because the components transform in such a way that the entire equation remains form-invariant under Lorentz transformations. For example,

$$A^{\mu} = B^{\mu} \quad \Rightarrow A'^{\mu} = \Lambda^{\mu}{}_{\nu}A^{\nu}, \quad B'^{\mu} = \Lambda^{\mu}{}_{\nu}B^{\nu}, \qquad A'^{\mu} = B'^{\mu}$$

5.3 Relativistic Mechanics

Example 55 (Relativistic Lagrangian of a Free Particle)**.** *We want to determine the action integral for a free material particle (a particle not under the influence of any external force). Note that this integral must be invariant under Lorentz transformations, then it follows that the action must depend on a scalar. Furthermore, it is clear that the*

integrand must be a dif- ferential of the first order. But the only scalar of this kind is the interval ds. So for a free particle the action must have the form

$$S = -\alpha \int_a^b ds,$$

where α is some constant.

Since $ds = c\gamma^{-1}\,dt$, the lagrangian is $L = -\alpha c\gamma^{-1}$. The quantity α, as already mentioned, characterizes the particle. In classical mechanics each particle is characterized by its mass m. Let us find the relation between a and m. It can be determined from the fact that in the limit as $c \to \infty$, our expression for L must go over into the classical expression $L = mv^2/2$. We find that $\alpha = mc$, the Lagrangian is

$$L = -mc^2\gamma^{-1},$$

and the action is

$$S = \int L\,dt = -mc \int ds.$$

Example 56 (Momentum four-vector)**.** *The four vector*

$$p_\mu = -\frac{\partial S}{\partial x^\mu} = mu_\mu = (\frac{E}{c}, p)$$

is the momentum four-vector.

For a free material particle,

- *The energy E of a free material particle is*

$$E = p \cdot v - L = \gamma mc^2.$$

- *The linear momentum is*

$$p = \frac{\partial L}{\partial v} = \gamma mv$$

Chapter 6

General Relativity

Example 57 (precession of the perihelion of Mercury)**.** *Notably, Newton's theory of gravity can not fully explain the precession of the perihelion of Mercury. The precession of Mercury's orbit is measured to be 5600 seconds of arc per century (one second of arc=1/3600 degrees), while Newton's theory predicts a precession of 5557 seconds of arc per century, which was out by 43"/century.*

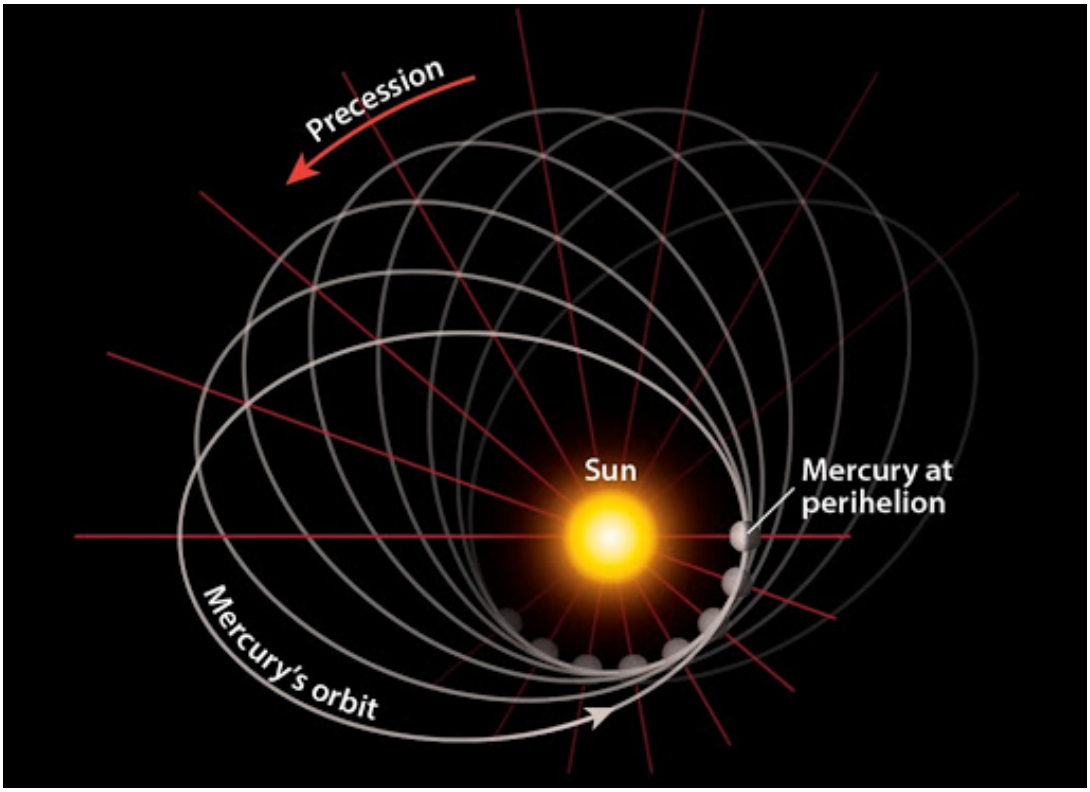

Figure 6.1: the precession of the perihelion of Mercury

Newton's theory of gravity is also incompatible with special relativity, which was published in 1905, in Einstein's paper "On the Electrodynamics of Moving Bodies". However just 10 years later, Einstein presented the final form of general relativity in his 1915 paper "The Field Equations of Gravitation".

6.1 Equivalence Principle

Remark 58 (the weak equivalence principle)**.** *In Newtonian gravity, the gravitational field is determined by Poisson's equation*

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 4\pi G\rho,$$

and particles move in gravity fields as

$$m_{inertial}a = \mathbf{F} = -m_{gravity}\nabla\phi = -m_{gravity}\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

It is a well-established experimental fact that $m_{inertial} = m_{gravity}$ for all particles, leading to the weak equivalence principle:

The inertial mass and gravitational mass of any object are equal.

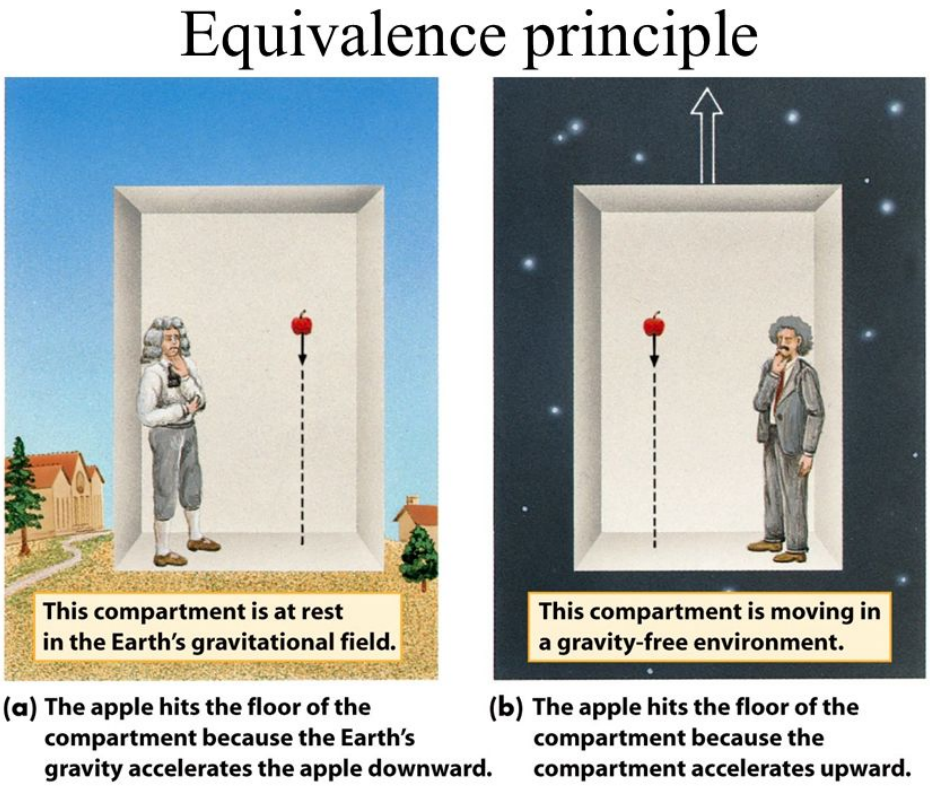

Figure 6.2: Einstein's elevator thought experiment

Theorem 59 (Einstein's equivalence principle)**.** *The weak equivalence principle holds, and in a free falling(non-rotating) laboratory occupying a small enough region of space-time, the laws of physics reduce to those of special relativity. It is impossible to detect the existence of a gravitational field by means of local experiments.*

TODO: why Ep leads to gravity=curvature? These ideas led Einstein to propose that gravity should no longer be regarded as a force in the conventional sense but rather as a manifestation of the curvature of the spacetime, this curvature being induced by the presence of matter.

Let (M, g) be a 4-dimensional manifold M representing the spacetime of the universe with a metric $g_{\mu\nu}$ which measures the spacetime interval $ds^2 = x^\mu x^\nu g_{\mu\nu}$. Denote $g = \det g_{\mu\nu}$.

Firstly, we shall show that a particle free falling under gravity moves along geodesic in the curved spacetime.

Theorem 60 (geodesic equations)**.** *If gravity is not regarded as a force but the curvature of spacetime, then the equations of motion of a particle moving only under the influence of gravity must be that of a 'free' particle in the curved spacetime.*

***Flat Spacetime**: We first consider the case of the flat spacetime.*

$$\frac{dp^\mu}{d\tau} = \frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right) = 0.$$

Since m (the rest mass) is constant, this simplifies to:

$$\frac{d^2x^\mu}{d\tau^2} = 0,$$

which is the geodesics in flat spacetime.

***Curved Spacetime**: In a curved spacetime, we need to account for the curvature:*

$$\frac{Du^\mu}{D\tau} = 0.$$

The covariant derivative $\frac{Du^\mu}{D\tau}$ in a coordinate basis is:

$$\frac{Du^\mu}{D\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta,$$

where $u^\alpha = \frac{dx^\alpha}{d\tau}$.

Setting the covariant derivative to zero for free-fall motion:

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0,$$

which explicitly gives:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

The above equation is known as the geodesic equation, describing the path of a free-falling particle in curved spacetime.

6.2 Field Equations

We shall first restrict our attention to a theory of gravity in vacuum.

Theorem 61 (Einstein field equations in vacuum, Einstein tensor, Einstein-Hilbert Action)**.**

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0,$$

here $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is called the Einstein tensor.

Equivalently, by contracting with $g^{\mu\nu}$ we find the Ricci scalar $R = 0$, so the equation reads

$$R_{\mu\nu} = 0.$$

This field equation can be derived via variational principle from the Einstein-Hilbert action:

$$S_{EH} = \int R\sqrt{-g}\, d^4x,$$

also define the Lagrangian $L_{EH} = R\sqrt{-g}$.

Theorem 62 (Einstein field equations in the presence of matter)**.**

$$G_{\mu\nu} = \kappa T_{\mu\nu},$$

here κ is the Einstein gravitational constant, related to the Newtonian gravitational constant G as $\kappa = \frac{8\pi G}{c^4}$.

This can be derived from the action

$$S = \frac{1}{2\kappa} S_{EH} + S_M = \int (\frac{1}{2\kappa} L_{EH} + L_M)\, d^4x.$$

where we note that the energy-momentum tensor has the property

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta L_M}{\delta g^{\mu\nu}}, \quad \delta S = \int \delta L\, d^4x.$$

For example, the electromagnetic field satisfies this relation:

$$T^{EM}_{\mu\nu} = -\frac{1}{\mu_0}(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}), \quad L_{EM} = -\frac{1}{4\mu_0}\int \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} F_{\nu\mu}\, d^4x.$$

6.3 Solutions to Einstein field Equation

Remark 63 (Flat Spacetime)**.** *The Minkowski metric $\eta_{\nu\mu}$, which describes flat spacetime in special relativity, is a solution to the Einstein field equations in vacuum.*

Example 64 (Spherical Mass, Schwarzschild metric)**.** *The Schwarzschild metric is an exact solution to the Einstein field equations that describes the gravitational field outside a spherical body of mass M, on the assumption that the electric charge of the mass, angular momentum of the mass are both zero.*

Einstein's field equations in a vacuum are given by: $G_{\mu\nu} = 0$,

We assume a static, spherically symmetric spacetime. Thus, the metric can be written as:

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{6.1}$$

where $\phi(r)$ and $\lambda(r)$ are functions to be determined.

Next, we calculate the Ricci tensor $R_{\mu\nu}$:

$$\begin{aligned}
R_{tt} &= e^{2(\phi-\lambda)}\left(\frac{d^2\phi}{dr^2} + \left(\frac{d\phi}{dr}\right)^2 - \frac{d\phi}{dr}\frac{d\lambda}{dr}\right),\\
R_{rr} &= -\frac{d^2\phi}{dr^2} - \left(\frac{d\phi}{dr}\right)^2 + \frac{d\phi}{dr}\frac{d\lambda}{dr} + \frac{2}{r}\frac{d\lambda}{dr},\\
R_{\theta\theta} &= 1 - e^{-2\lambda}\left(1 - r\frac{d\lambda}{dr}\right),\\
R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta}.
\end{aligned}$$

Substituting these into the vacuum field equations $R_{\mu\nu} = 0$, we obtain a set of differential equations:

$$\frac{d^2\phi}{dr^2} + \left(\frac{d\phi}{dr}\right)^2 - \frac{d\phi}{dr}\frac{d\lambda}{dr} = 0,$$
$$\frac{d^2\lambda}{dr^2} + \left(\frac{d\lambda}{dr}\right)^2 - \frac{d\phi}{dr}\frac{d\lambda}{dr} = 0,$$
$$e^{2\lambda}\left(1 - \frac{d\lambda}{dr}\right) = 1.$$

Solving the differential equations, integrating, we find:

$$e^{2\lambda(r)} = \left(1 - \frac{2GM}{r}\right)^{-1}, \tag{6.2}$$

and

$$e^{2\phi(r)} = 1 - \frac{2GM}{r}. \tag{6.3}$$

Substituting these solutions back into the metric, we get the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{6.4}$$

Here $r_s := 2GM$ is called the Schwarzschild radius of the massive body.

Remark 65 (Birkhoff's theorem)**.** *Based on the derivation above, the Schwarzschild solution is the unique spherically symmetric asymptotically flat solution to the Einstein field equations in vacuum.*

Remark 66. *Note that the Schwarzschild metric solves the Einstein field equation in vacuum, so the metric is only valid from infinity down to the outside surface of the spherical body, such as a star or a planet. For most objects, the solution is valid since the Schwarzschild radius is much smaller than the object's physical radius. Next, we discuss objects whose physical radius is smaller than its Schwarzschild radius.*

Definition 67 (Event Horizon)**.** *An event horizon is a boundary beyond which events cannot affect an outside observer.*

An event horizon in the context of a spacetime manifold M can be defined mathematically as follows:

Let $\mathcal{I}^+$ denote future null infinity(the "destination" of light rays (null geodesics) that escape to infinity as time progresses) and $\mathcal{S} \subset M$ the set of points in the spacetime manifold that can send light signals to $\mathcal{I}^+$. The event horizon $\mathcal{H}$ is the boundary of the set $\mathcal{S}$:

$$\mathcal{H} = \partial\mathcal{S}.$$

The event horizon is a null surface, a hypersurface in spacetime where the normal vector is null (light-like).

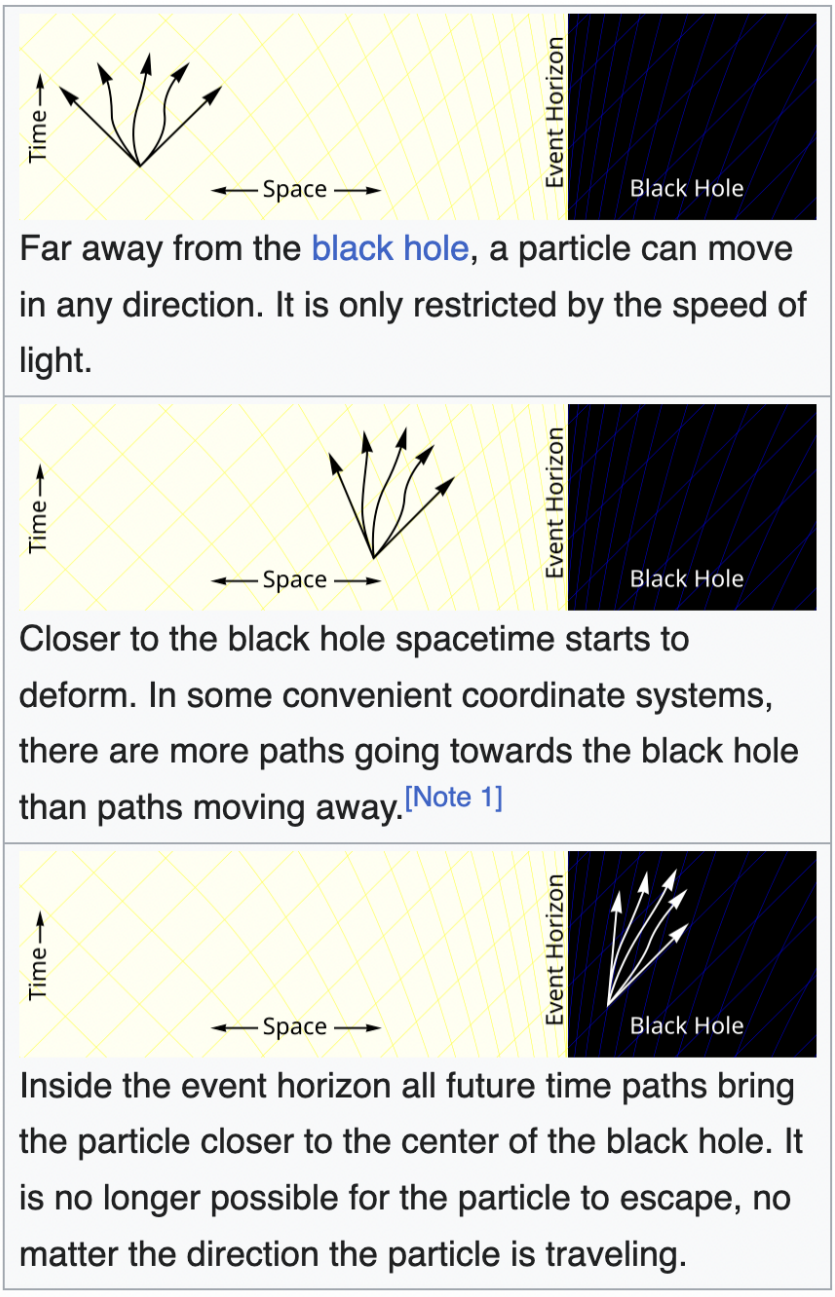

Figure 6.3: event horizon of a black hole

Example 68 (Schwarzschild Black Hole)**.** *The Schwarzschild solution appears to have singularities at $r = 0$ and $r = r_s$. $r = 0$ is an intrinsic curvature singularity. We now focus on the singularity r_s at event horizon, which arises from our choice of coordinates. This singularity becomes regular in some other coordinate systems.*

We show that light cannot escape from within r_s.

For radial light paths (light moving directly outward or inward, so $d\theta = 0$ and $d\phi = 0$):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0.$$

Simplify to find the relationship between dt and dr:

$$\frac{dr}{dt} = \pm c \left(1 - \frac{2GM}{r}\right),$$

where $\pm$ accounts for the possibility of inward or outward motion.

At the Event Horizon, $r = r_s$ Schwarzschild radius, $\frac{dr}{dt} = 0$.

This means that at the event horizon, the radial coordinate r for light does not change with respect to time. Light emitted radially from this boundary is trapped, unable to move outward or inward.

Inside the Event Horizon $r < r_s$, the term $\left(1 - \frac{2GM}{r}\right)$ becomes negative, leading to an imaginary result for the velocity $\frac{dr}{dt}$, which is nonphysical. This indicates that no real paths exist for light that allow it to escape once within the event horizon.

- $\mathbf{r > r_s}$*: This describes the region outside the event horizon of a Schwarzschild black hole. Here, objects and light can still escape the gravitational pull of the black hole.*
- $\mathbf{r = r_s}$*: This describes the event horizon itself, which is the boundary at which the escape velocity equals the speed of light. Beyond this point, nothing can escape the gravitational pull, not even light.*
- $\mathbf{r < r_s}$*: This describes the region inside the event horizon. In this region, all paths lead towards the singularity at the center of the black hole, and nothing can escape.*

Remark 69 (Gravitational Collapse, White Dwarf, Neutron Star)**.** *The black hole can be formed via gravitational collapse.*

A star is held up by a mixture of gas and radiation pressure. The energy to provide this pressure support comes from nuclear fusion, mainly hydrogen to helium. When all the nuclear fuel is used up, the star begins to cool and collapse under its own gravity.

Denote the sun's mass by $M_\odot$. After collapse, the star has three destinies based on its final mass:

- *Star mass $< 1.4M_\odot$: White Dwarf, consist mainly of electron-degenerate matter, typically composed of carbon and oxygen, held up by electron degeneracy pressure.*
- *$1.4M_\odot <$ Star mass $< 3M_\odot$: Neutron Star. The extreme densities cause electrons and protons to merge into neutrons through inverse beta decay. Its pressure is supported by degenerate neutrons.*
- *Star mass $> 3M_\odot$: Black Hole*

Example 70 (Wormhole and Einstein-Rosen Bridge)**.** *As of now, neither wormholes nor Einstein-Rosen bridges have been physically detected.*

TODO

Example 71 (Charged Mass, Reissner–Nordström metric)**.** *Most astrophysical black holes are generally believed to be uncharged.*

Nonetheless, now we study the solution outside a static spherically symmetric charged body. The exterior is now not a vacuum but filled with a static electric field. In this case the solution shall satisfy the Einstein equations, which take the form

$$R_{\mu\nu} = -\kappa T^{EM}_{\mu\nu},$$

and the Maxwell equations.

In spherical coordinates (t, r, θ, φ), the Reissner–Nordström metric (i.e., the line element) is given by

$$ds^2 = c^2\,d\tau^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2\,dt^2 - \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 - r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2),$$

where t is the time coordinate (measured by a stationary clock at infinity), r_Q is a characteristic length scale given by $r_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4}$, Q is the total charge of the object, ε_0 is the electric constant.

Example 72 (Rotating Mass, Kerr metric, Penrose Process)**.** *Observational evidence suggests that most black holes rotate. The Kerr metric describes the geometry of empty spacetime around a rotating uncharged axially symmetric black hole.*

Let J represent the angular momentum of the rotating black hole. The metric (or equivalently its line element for proper time) in Boyer–Lindquist coordinates is given by:

$$\begin{aligned} ds^2 &= -c^2 d\tau^2 \\ &= -\left(1 - \frac{r_s r}{\Sigma}\right) c^2 dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\,d\varphi^2 - \frac{2 r_s r a \sin^2\theta}{\Sigma} c\,dt\,d\varphi, \end{aligned}$$

where the coordinates (r, θ, φ) are standard oblate spheroidal coordinates,
and where for brevity, the length scales a, Σ, and Δ have been introduced as:

$$a = \frac{J}{Mc},$$

$$\Sigma = r^2 + a^2\cos^2\theta,$$

$$\Delta = r^2 - r_s r + a^2.$$

The Kerr black hole has two characteristic surfaces: the event horizon at $\Delta = 0$, which marks the boundary beyond which nothing can escape, and the ergosphere, an oblate region outside the horizon where no observer can remain at rest due to frame dragging.

The Penrose process allows energy to be extracted from a rotating black hole by having a particle split inside the ergosphere, where one fragment falls into the hole with negative energy while the other escapes to infinity with more energy than the original particle. This mechanism gradually reduces the black hole's angular momentum.

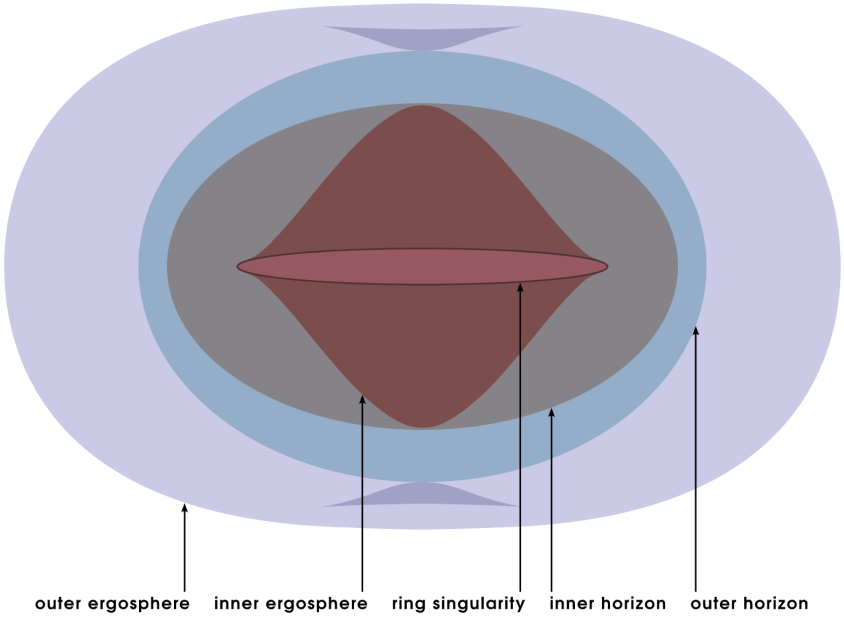

Figure 6.4: Kerr Black Hole

Example 73 (Charged and Rotating Mass, Kerr–Newman metric)**.** *The Kerr–Newman metric describes the spacetime geometry around a mass which is electrically charged and rotating. In Boyer–Lindquist coordinates, the metric can be written in the form:*

$$c^2 d\tau^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + \left(c\,dt - a\sin^2\theta\,d\varphi\right)^2\frac{\Delta}{\rho^2} - \left(\left(r^2+a^2\right)d\varphi - ac\,dt\right)^2\frac{\sin^2\theta}{\rho^2},$$

where the coordinates (r, θ, φ) are standard spherical coordinates, and the length scales are given by:

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

$$\Delta = r^2 - r_s r + a^2 + r_Q^2,$$

with r_s being the Schwarzschild radius, r_Q being the characteristic length scale for charge, and a being the Kerr parameter.

6.4 Symmetry

TODO

Remark 74 (Diffeomorphism)**.**

Definition 75 (Killing Vector)**.**

6.5 Black Hole Thermodynamics

TODO

Theorem 76 (Hawking radiation, area)**.**

Definition 77 (temperature, entropy)**.**

6.6 Gravitational Waves

The field equations are hard to solve exactly. Assume that the gravitational field is weak, so we can linearize it as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h| << 1,$$

also define the trace $h = h^\mu_\mu$ and $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.

Under the Lorentz gauge $\partial_\nu \bar{h}^{\mu\nu} = 0$, the linearized field equations can be simplified as

$$\partial^\mu \partial_\mu \bar{h}^{\mu\nu} = -2\kappa T^{\mu\nu}.$$

The linearized field equations admit plane-wave solutions of the form

$$\bar{h}^{\mu\nu} = A^{\mu\nu} e^{ik_\rho x^\rho},$$

and physical solutions may be obtained by taking the real part of it.

Since the vacuum field equations are linear, the general solution can be written as a superposition of plane-wave solutions

$$\bar{h}^{\mu\nu} = \int A^{\mu\nu}(\mathbf{k}) e^{ik_\rho x^\rho} \, d^3\mathbf{k}, \quad k^\rho = (k^0, \mathbf{k}).$$

Example 78 (polarization of gravitational waves)**.** *Assume a plane gravitational wave propagates along x^3 direction $\bar{h}^{\mu\nu} = A^{\mu\nu} e^{ik_\rho x^\rho}$, $k^\rho = (k, 0, 0, k)$.*

Besides the Lorentz gauge, we further choose the Transverse-Traceless gauge, where the transverse condition gives $\partial^i h_{ij} = 0$, $h_{0\mu} = 0$, and the traceless condition gives $h = 0$. Now

$$A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^+ & h^\times & 0 \\ 0 & h^\times & -h^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h^+ + h^\times,$$

representing a superposition of two linear polarizations, just like the plane-wave solutions of electromagnetic waves.

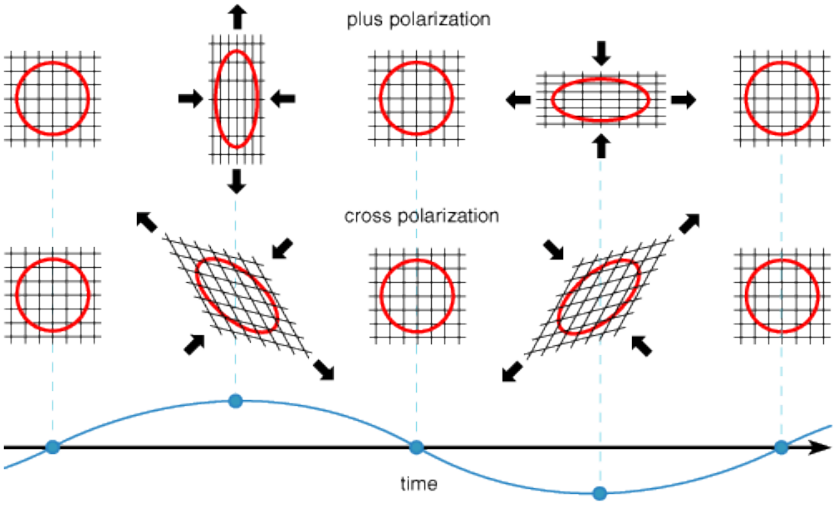

Figure 6.5: oscillatory deformation due to gravitational waves

- *h^+ polarization: Along the x-axis, particles are stretched; along the y-axis, particles are squeezed. This leads to an oscillatory deformation of a ring of test masses into an ellipse that alternates between vertical and horizontal stretching.*
- *$h^\times$ polarization: This distorts the ring into an ellipse rotated by 45° compared to the case. The distortion oscillates diagonally, squeezing along the x+y and x-y directions.*

Theorem 79 (energy momentum tensor of gravitational waves)**.** *In the linearized theory of gravity, to include the contribution of the energy-momentum associated with the gravitational field itself, we modify the field equations to read*

$$G^{(1)}_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} + t_{\mu\nu}),$$

where $G^{(i)}_{\mu\nu}$ is the i-th order Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor of the matter present and $t_{\mu\nu}$ is the energy-momentum tensor of the gravitational field.

$$t_{\mu\nu} \approx \frac{c^4}{8\pi G} G^{(2)}_{\mu\nu} = \frac{c^4}{32\pi G} < \partial_\mu h_{ij} \partial_\nu h^{ij} > .$$

Example 80 (Energy Loss due to Gravitational Wave Emission)**.** *Since gravitational waves carry away energy, we expect energy to be lost from the physical system generating the gravitational radiation. The total energy (luminosity) carried away radiated away by gravitational waves is given by the quadrupole formula*

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5}\left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle,$$

where Q_{ij} is the mass quadrupole moment of the system, $\dddot{Q}_{ij}$ is its third time derivative (rate of change of acceleration).

In the multipole expansion of the wave equation, the leading-order radiating term is the quadrupole moment Q_{ij}, since mass-energy conservation forbids monopole radiation, and momentum conservation cancels dipole radiation.

We omit the details.

6.7 Cosmology

TODO No plan to study it right now.

Example 81 (Friedmann-Robertson-Walker (FRW) metric)**.** *The Friedmann-Robertson-Walker (FRW) metric is a solution to Einstein's field equations in general relativity that describes a homogeneous, isotropic expanding or contracting universe.*

Part III

Thermal Physics

Chapter 7

Thermodynamics

7.1 Thermal Properties

We first list the laws of thermodynamics.

- Zeroth Law: If two systems are in thermal equilibrium with a third system, then they must be in thermal equilibrium with each other. In other words, temperature $\tau_1 = \tau_3, \tau_2 = \tau_3 \rightarrow \tau_1 = \tau_2$.
- First Law(1860): Heat is a form of energy. (Conservation of Energy)
- Second Law(1824): Law of Increase of Entropy.
- Third Law(1906-1912): The entropy of the system approaches a constant value as the temperature approaches zero.(With the exception of glasses.)

We develop thermal physics from the concept of stationary quantum states of a system of particles. The entropy is the logarithm of the number of states. The dependence of the entropy on the energy of the system defines the temperature. From the entropy, the temperature, and the free energy we find the pressure, the chemical potential, and all other thermodynamic properties of the system.

Example 82 (Binary Model System, multiplicity function, fundamental assumption of thermal physics, closed system, accessible state, thermal contact, most probable configuration, sharp, thermal equilibrium)**.** *Consider N magnets each with magnetic moment m, with $N_+ = \frac{1}{2}N + s$ magnets up and $N_- = \frac{1}{2}N - s$ magnets down. The net magnetic moment is $2sm$. The potential energy of the system in a uniform magnetic field B is $U = -2sm \cdot B$.*

In general, we write the number of states of a system, called the multiplicity function, with N particles and energy U as $g(N, U)$.

The fundamental assumption of thermal physics is that a closed system (constant energy, constant number of particles, constant volume, constant electromagnetic field, etc) is equally likely to be in any of the quantum states accessible (energy, number of particles of the state are the same as those of the system) to it.

If two systems are bought together so that energy can be transferred freely from one to the other, this is called thermal contact.

The multiplicity of the combined system is

$$g(N, U) = \sum_{U_1} g_1(N_1, U_1) g_2(N_2, U - U_1).$$

For a certain $\hat{U}_1$, the quantity $g_1(N_1, \hat{U}_1)g_2(N_2, U - \hat{U}_1)$ achieves its maximum, and we call this the most probable configuration. If the system is large, the maximum with respect to changes in U_1 will be extremely sharp (which is believed to be a general property of all large systems), so the fluctuations around the most probable configuration are small. Therefore, the average physical properties of a large system over all accessible states may be replaced by the averages over only the most probable configurations. Such averages are called thermal equilibrium values.

Definition 83 (Entropy, Temperature)**.** *To find the most probable configuration, we must have (note that $U_1 + U_2 = U$)*

$$\frac{\partial g_1}{\partial U_1} g_2 + g_1 \frac{\partial g_2}{\partial U_1} = 0, \quad or, \quad \frac{1}{g_1}\frac{\partial g_1}{\partial U_1} = \frac{1}{g_2}\frac{\partial g_2}{\partial U_2} \quad or, \quad \frac{\partial \log g_1}{\partial U_1} = \frac{\partial \log g_2}{\partial U_2}$$

The entropy σ is

$$\sigma(N, U) = \log g(N, U)$$

In thermal equilibrium the temperatures of the two systems are equal.

The fundamental temperature τ and the Kelvin temperature T are defined as

$$\frac{1}{\tau} = \frac{\partial \sigma}{\partial U}, \quad \tau = k_B T.$$

For two systems in thermal contact, energy flows from higher temperature system to the lower until they have the same temperature.

The conventional entropy S is defined as $\frac{1}{T} = \frac{\partial S}{\partial U}, \quad S = k_B \sigma$.

We see that thermal contact between two systems tends to increase the number of total accessible states, or the entropy.

Theorem 84 (Law of Increase of Entropy)**.** *The entropy of a closed system tends to remain constant or to increase when a constraint internal to the system is removed.*

Definition 85 (Boltzmann factor, partition function, reversible process, pressure)**.** *We want to find the probability of a system in a state s of energy E_s, when this system is in thermal contact with a large reservoir at temperature τ. Say the total energy is U_0.*

The reservoir energy is $U_0 - E_s$. The multiplicity of the whole, since we specify the system is in state s, is just the multiplicity of the reservoir R,

$$g_R(U_0 - E_s) = \exp(\sigma_R(U_0 - E_s)) \approx \exp(\sigma_R(U_0) - E_s \frac{\partial \sigma_R}{\partial U}) = \exp(\sigma_R(U_0) - \frac{E_s}{\tau}),$$

where we assume $E_s << U_0$.

The Boltzmann factor is $\exp(-\frac{E}{\tau})$.

The partition function is

$$Z = \sum_s \exp(-\frac{E_s}{\tau}),$$

so the probability of system in state s is $P(E_s) = \frac{\exp(-E_s/\tau)}{Z}$, and the average energy of the system $U := E_s[U_{system}] = \sum_s P(E_s)E_s = \tau^2 \frac{\partial \log Z}{\partial \tau}$.

A reversible process is a process, involving a system and its surroundings, whose direction can be reversed by infinitesimal changes in some properties of the surroundings, such as pressure or temperature.

Now consider a reversible process, where we change the volume slowly such that the state remains in the same quantum state s, but the associated energy E_s slowly changes as the volume changes, which means the energy E_s is a function of the volume of the system.

For a state s undergoing a reversible small volume change δV, its energy change is

$$E_s(V - \delta V) = E_s(V) - \frac{\partial E_s}{\partial V}\delta V.$$

The mechanical work done on the system by the pressure in a contraction of volume is equal to the change of energy of the system.

The pressure p, averaged over all state s, is

$$p = -\frac{\partial U}{\partial V} = \tau \frac{\partial \sigma}{\partial V}.$$

Definition 86 (diffusive contact, diffusive equilibrium, chemical potential, thermodynamic identity, Gibbs factor, Gibbs sum, absolute activity)**.** *In thermal contact, two systems exchange energy to reach same temperature.*

Next we consider two systems that can exchange particles in diffusive contact, where molecules can move by diffusion through a permeable interface. If the net particle flow is zero, we say the systems are in diffusive equilibrium.

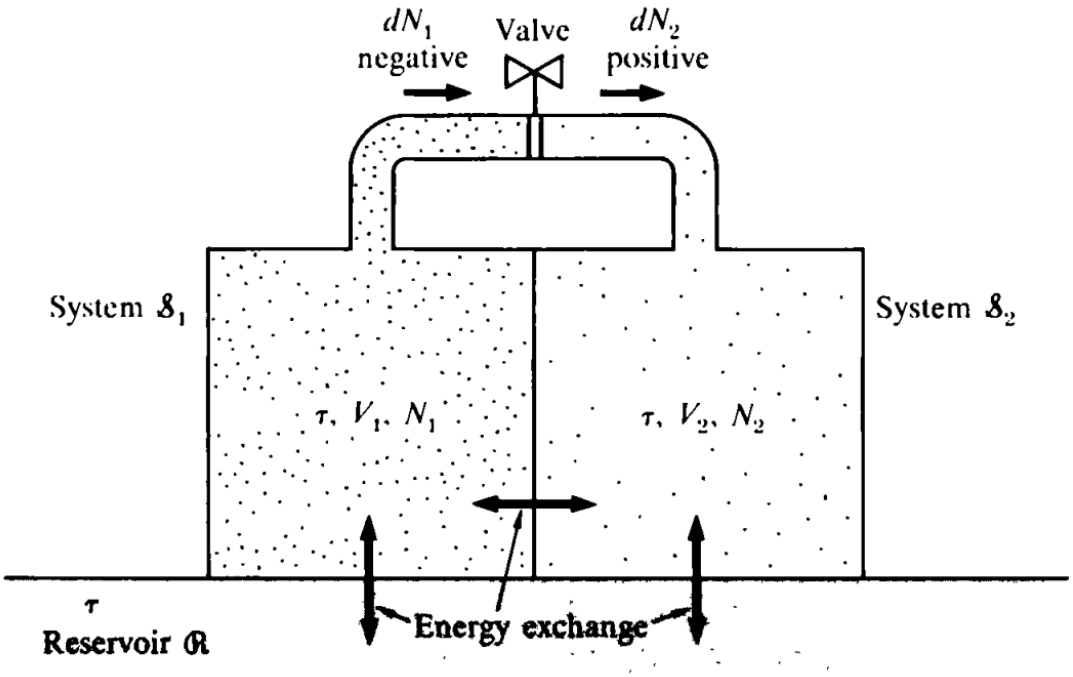

Figure 5.1 Example of two systems, $\mathcal{S}_1$ and $\mathcal{S}_2$, in thermal contact with each other and with a large reservoir $\mathcal{R}$, forming a closed total system. By opening the valve, $\mathcal{S}_1$ and $\mathcal{S}_2$ can be brought in diffusive contact while remaining at the common temperature τ. The arrows at the valve have been drawn for a net particle transfer from $\mathcal{S}_1$ to $\mathcal{S}_2$.

Figure 7.1: Diffusive Contact

Particles flow from the system with the higher chemical potential to the lower. (e.g. battery)

For two systems S_1, S_2 in thermal and diffusive equilibrium, its Helmholtz free energy is a minimum.

$$0 = dF = \frac{\partial F_1}{\partial N_1} dN_1 + \frac{\partial F_2}{\partial N_2} dN_2 \to \frac{\partial F_1}{\partial N_1} = \frac{\partial F_2}{\partial N_2}.$$

The chemical potential at temperature τ, volume V and number of particles N (note that particles are indivisible) is related to the Helmholtz energy F by

$$\mu(\tau, V, N) = F(\tau, V, N) - F(\tau, V, N-1) \approx \frac{\partial F}{\partial N} = -\tau \frac{\partial \sigma}{\partial N}.$$

The Boltzmann factor $\exp(-E_s/\tau)$ relates to the probability that a system be in a certain state s of energy E_s when it is in thermal contact with a reservoir of temperature τ.

The Gibbs factor generalizes the Boltzmann factor to a system in contact with a reservoir of temperature τ and chemical potential μ.

$$P(N_s, E_s) = \exp(\sigma(N_0 - N_s, U_0 - E_s)) \approx \exp(\sigma(N_0, U_0) + N_s \frac{\mu}{\tau} - E_s \frac{1}{\tau})$$

The Gibbs factor is

$$\exp(\frac{N_s \mu - E_s}{\tau}).$$

The Gibbs sum / grand partition function is

$$Z(\mu, \tau) = \sum_{N=0}^{\infty} \sum_{s \in S(N)} \exp(\frac{N\mu - E_s}{\tau}).$$

The probability that a system is in a state with N particles and energy E is

$$P(N, E) = \frac{\exp(\frac{N\mu - E}{\tau})}{Z(\mu, \tau)}.$$

The absolute activity is $\lambda = \exp(\mu/\tau)$ and the average number of particles is $< N >= \lambda \frac{\partial \log Z}{\partial \lambda}$.

Definition 87 (the thermodynamic identity, heat, work, Helmholtz free energy, Gibbs free energy, chemical work)**.** *Now suppose the number of states/entropy of the system depends only on potential energy U and volume V, for a fixed number of particles, so the differential of the entropy is:*

$$d\sigma(U, V) = \frac{1}{\tau} dU + \frac{p}{\tau} dV, \quad \tau d\sigma = dU + pdV,$$

which we call the thermodynamic identity. Equivalently,

$$dU = \tau d\sigma - pdV,$$

where we view $dQ = \tau d\sigma$ as the heat (transfer of energy by thermal contact with a reservoir) added to the system and $-pdV$ as the work (transfer of energy by a change

in the external parameters of the system that does not change the entropy in a reversible process) done on the system by volume change.

The work performed on a system at constant temperature in a reversible process is the change in Helmholtz free energy. The Helmholtz free energy is

$$F = U - \tau\sigma = -\tau \log Z.$$

It balances minimum energy and maximum entropy, and is a minimum for a system in thermal contact with reservoir R if the volume of the system is constant.

Some useful formulas:

$$\sigma = -\frac{\partial F}{\partial \tau}, \quad p = -\frac{\partial F}{\partial V}$$

Allowing the number of particles to change, the thermodynamic identity becomes

$$dU = \tau d\sigma - pdV + \mu dN.$$

The work performed on a system at constant temperature and pressure in a reversible process is the change in Gibbs free energy. The Gibbs free energy is

$$G = U - \tau\sigma + pV.$$

The differential is

$$dG = \mu dN - \sigma d\tau + V dp$$

The variables τ and p are intensive quantities: they do not change value when two identical systems are put together. But U, σ, V, and G are linear in the number of particles N, so they are extensive quantities.

$$G(N, p\tau) = N\mu(p, \tau).$$

The chemical work performed on a system in the reversible process that transfers dN particles to the system is $\mu\, dN$.

Example 88 (Law of Mass Action)**.** *A chemical reaction, such as $H_2 + Cl_2 - 2HCl = 0$, can be written as $\sum v_i A_i = 0$, where A are the chemical species. Assume the reaction is held at constant pressure and temperature. The Gibbs free energy is a minimum at equilibrium:*

$$0 = dG = \sum \mu_j \, dN_j \propto \sum v_j \mu_j.$$

Let n_j, Z_j, F_j be the concentration, internal partition function, internal free energy of species j, then for ideal gas a useful result is $\mu_j = \tau(\log n_j - \log(n_{Qj} Z_j))$.

The law of mass action is

$$\prod_j n_j^{v_j} = K(\tau) := \prod_j n_{Qj}^{v_j} \exp(-v_j F_j/\tau),$$

where $K(\tau)$ is the equilibrium constant.

The law of mass action says that the product of the concentrations of the reactions is a function of the temperature alone.

7.2 Gas

We first discuss the ideal gas.

Example 89 (orbital, distribution function, ideal gas, Fermi-Dirac distribution function, Fermi level, Fermi energy, Bose-Einstein distribution function)**.** *An orbital is a state of the Schrodinger equation for only one particle. If the interactions between particles are weak, the orbital model allows us to approximate an exact quantum state of the Schrodinger equation of a system of N particles in terms of an approximate quantum state that we construct by assigning the N particles to orbitals, with each orbital a solution of a one-particle Schrodinger equation. There are usually an infinite number of orbitals available for occupancy. The orbital model gives an exact solution of the N-particle problem only if there are no interactions between the particles.*

The thermal average value of the number of particles that occupy an orbital(thermal average occupancy) is called the distribution function, usually designated as $f(E, \tau, \mu)$, where E is the energy of the orbital.

In quantum mechanics, all species of particles fall into two distinct classes, fermions and bosons. Any particle with half-integral spin is a fermion, and any particle with zero or integral spin is a boson. Composite particles follow the same rule: an atom of ^3He is composed of an odd number of particles-2 electrons, 2 protons, 1 neutron, each of spin 1/2, so that ^3He must have half-integral spin and must be a fermion. An atom of ^4He has one more neutron, so ^4He must be a boson.

The fermion or boson nature of the particle species that make up a many-body system has a profound and important effect on the states of the system. The orbital model of noninteracting particles satisfies the occupancy rules:

- *An orbital can be occupied by any integral number of bosons of the same species, including zero.*
- *(Pauli exclusion principle) An orbital can be occupied by O or 1 fermion of the same species.*

So for bosons, the Gibbs sum is over all positive integers of the orbital occupancy N, while for fermions N=0,1 only. When $f(E, \tau, \mu) << 1$, the occupancies N=2,3,... do not matter, and thus the boson and fermion distribution functions are similar. This is the classical regime.

Ideal gas is a gas of non-interacting atoms in the classical regime. The total energy $E_{tot} = \sum e_n N_n$, where e_n, N_n are the energy, the number of particles in orbital n respectively.

Consider a system composed of a single orbital that may be occupied by a fermion. Say the energy of the system is zero if the orbital is unoccupied and is e if the orbital is occupied by one fermion. The Gibbs sum is

$$Z = 1 + \lambda \exp(-e/\tau).$$

The Fermi-Dirac distribution function, which equals the thermal average value of the occupancy of the orbital, is

$$f(e) =< N >= \frac{(0 * 1 + 1 * \lambda \exp(-e/\tau))}{Z} = \frac{1}{\exp((e-\mu)/\tau) + 1}.$$

The Fermi level is the chemical potential μ.

The Fermi energy is the chemical potential at zero temperature, $e_F = \mu(\tau = 0)$.

Consider the distribution function for a system of noninteracting bosons in thermal and diffusive contact with a reservoir. We assume the bosons are all of the same species. Let e denote the energy of a single orbital when occupied by one particle; when there are N particles in the orbital, the energy is Ne. The Gibbs sum is

$$Z = \sum_{N=0}^{\infty} (\lambda \exp(-e/\tau))^N = \frac{1}{1 - \lambda \exp(-e/\tau)}.$$

The Bose-Einstein distribution function, which is the thermal average of the number of particles in the orbital, is

$$f(e) = \lambda \frac{\partial \log Z}{\partial \lambda} = \frac{1}{\exp((e - \mu)/\tau) - 1}$$

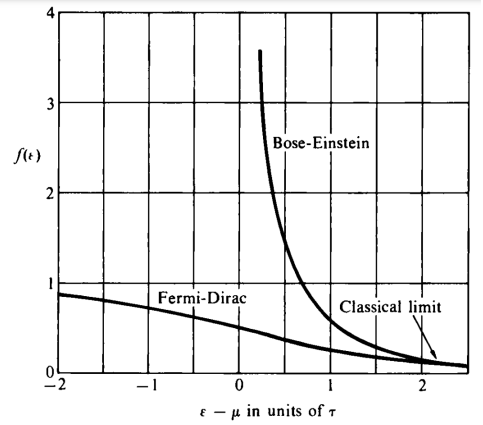

Figure 6.6 Comparison of Bose-Einstein and Fermi-Dirac distribution functions. The classical regime is attained for $(\varepsilon - \mu) \gg \tau$, where the two distributions become nearly identical. We shall see in Chapter 7 that in the degenerate regime at low temperature the chemical potential μ for a FD distribution is positive, and changes to negative at high temperature.

Figure 7.2: distribution functions

Now we consider the ideal gas.

By definition, a gas is in the classical regime when the average number of atoms in each orbital is much less than one. The average orbital occupancy for a gas at room temperature and atmospheric pressure is of the order of only 10^{-6}, safely in the classical regime. In the classical regime, the classical distribution function for both bosons and fermions is approximately

$$f(e) \approx \lambda \exp(-e/\tau).$$

Consider one (N=1) atom of mass M in a box of volume V, the concentration $n = \frac{N}{V} = \frac{1}{V}$. From quantum mechanics we can get its orbitals and energy levels. Its partition function is $Z = n_Q/n$, where the quantum concentration is

$$n_Q = (\frac{M\tau}{2\pi\hbar^2})^{3/2}.$$

When $n << n_Q$ the gas is in classical regime. Note that $\lambda = \exp(\mu/\tau) = n/n_Q$.

The chemical potential is $\mu = \tau \log(n/n_Q)$.
The free energy, since $\frac{\partial F}{\partial N} = \mu$, *is*

$$F = \int_0^N \mu(N, \tau, V)\, dN = N\tau(\log(n/n_Q) - 1).$$

The pressure $p = -\frac{\partial F}{\partial V} = \frac{N\tau}{V}$. *The ideal gas equation is*

$$pV = N\tau.$$

The thermal energy $U = F + \tau\sigma = \frac{3}{2}N\tau$.
The entropy is $\sigma = -\frac{\partial F}{\partial \tau} = N(\log(n_Q/n) + \frac{5}{2})$.

Example 90 (quantum gas, degenerate Fermi gases, Einstein condensation)**.** *When* $n >> n_Q$ *the gas is called quantum gas, where fermions and bosons behave differently.*
TODO
A Fermi gas is called degenerate when the temperature is low in comparison with the Fermi energy. The important applications of the theory of degenerate Fermi gases include conduction electrons in metals; the white dwarf stars; liquid ^3He*; and nuclear matter. The most striking property of a fermion gas is the high kinetic energy of the ground state of the system at absolute zero.*
In a gas of noninteracting bosons at a certain transition temperature, below which a substantial fraction of the total number of particles in the system will occupy the single orbital of lowest energy, called the ground orbital. The ground-orbital effect is called the Einstein condensation. The Einstein condensation temperature for a gas of fictitious noninteracting helium atoms at the observed density of liquid helium is very much higher, about 3 K.

7.3 Kinetic Theory

Example 91 (Maxwell velocity distribution)**.** *The Maxwell velocity distribution describes the statistical distribution of velocities for particles in a gas at thermal equilibrium. For a gas at temperature* T, *the probability density function for a particle to have velocity* $\vec{v}$ *is given by:*

$$f(\vec{v}) = \left(\frac{m}{2\pi\tau}\right)^{3/2} \exp\left(-\frac{m|\vec{v}|^2}{2\tau}\right)$$

where m *is the particle mass, and* $|\vec{v}|$ *is the magnitude of the velocity. This distribution follows directly from the principles of statistical mechanics when applying the Boltzmann distribution to the kinetic energy of gas particles.*

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Example 92 (Transport Phenomena in Materials)**.** *Transport phenomena describe the exchange of mass, energy, and momentum within physical systems. Three fundamental transport processes are:*

Diffusion: *The movement of particles from regions of high concentration to regions of low concentration due to random molecular motion. Mathematically described by Fick's laws:*

- ***Fick's first law:*** *The diffusive flux $\vec{J}$ is proportional to the negative gradient of concentration ∇c:*
$$\vec{J} = -D\nabla c$$
where D is the diffusion coefficient $[m^2/s]$.

- ***Fick's second law:*** *Describes the rate of change of concentration:*
$$\frac{\partial c}{\partial t} = D\nabla^2 c$$

Thermal Conductivity: *The property of a material that indicates its ability to conduct heat. It appears in Fourier's law of heat conduction:*

$$\vec{q} = -k\nabla T$$

where $\vec{q}$ is the heat flux $[W/m^2]$, k is the thermal conductivity $[W/(m \cdot K)]$, and ∇T is the temperature gradient $[K/m]$. The thermal diffusivity $\alpha = k/(\rho c_p)$ $[m^2/s]$ relates thermal conductivity to density ρ and specific heat capacity c_p.

Viscosity: *A measure of a fluid's resistance to deformation by either shear stress or tensile stress. For Newtonian fluids, the shear stress τ is proportional to the velocity gradient:*

$$\tau = \mu \frac{du}{dy}$$

where μ is the dynamic viscosity $[Pa \cdot s]$ and du/dy is the velocity gradient perpendicular to the direction of shear. The kinematic viscosity $\nu = \mu/\rho$ $[m^2/s]$ is the ratio of dynamic viscosity to density.

These transport coefficients are all related to microscopic molecular motion and, in gases, share similar temperature dependencies as predicted by kinetic theory.

Example 93 (Detailed Balance)**.** *At equilibrium, each elementary process and its reverse occur at exactly the same rate.*

Mathematical Definition: *For a system with states i and j, detailed balance requires:*

$$P_i W_{i \to j} = P_j W_{j \to i}$$

where P_i is the probability of being in state i, and $W_{i \to j}$ is the transition rate from state i to state j.

Microscopic Reversibility: *Detailed balance is a manifestation of microscopic reversibility, reflecting the time-reversal symmetry of fundamental physical laws. For systems in thermal equilibrium at temperature T, the transition rates satisfy:*

$$\frac{W_{i \to j}}{W_{j \to i}} = e^{-(E_j - E_i)/k_B T}$$

where E_i and E_j are the energies of states i and j, and k_B is Boltzmann's constant.

Example 94 (Heat Conduction Equation)**.** *The heat conduction equation, also known as the heat diffusion equation or Fourier-Biot equation, is a partial differential equation that describes how temperature distributes itself over time in a spatial domain due to thermal conduction.*

Differential Form: *The general form of the heat equation in three dimensions is:*

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{q}{\rho c_p}$$

where:

- $T(\mathbf{r}, t)$ *is the temperature field as a function of position* $\mathbf{r}$ *and time* t
- $\alpha = \frac{k}{\rho c_p}$ *is the thermal diffusivity* $[m^2/s]$
- k *is the thermal conductivity* $[W/(m \cdot K)]$
- ρ *is the density* $[kg/m^3]$
- c_p *is the specific heat capacity at constant pressure* $[J/(kg \cdot K)]$
- $q(\mathbf{r}, t)$ *is the volumetric heat source term* $[W/m^3]$
- ∇^2 *is the Laplacian operator:* $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ *in Cartesian coordinates*

Physical Interpretation: *The equation states that the rate of change of temperature at a point is proportional to the curvature of the temperature field (the Laplacian) plus any internal heat generation. The thermal diffusivity* α *determines how quickly heat spreads through the material.*

Derivation: *The heat equation is derived from two fundamental principles:*

1. ***Fourier's Law of Heat Conduction:*** *The heat flux is proportional to the negative temperature gradient:*

$$\mathbf{q} = -k \nabla T$$

2. ***Conservation of Energy:*** *Applied to a differential volume element, yielding:*

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + q$$

Combining these principles gives the heat conduction equation.

TODO

Chapter 8

Phase Transitions

Definition 95 (phase, isotherm, critical temperature, the Clausius–Clapeyron relation, enthalpy, latent heat of vaporization)**.** *A phase is a portion of a system that is uniform in composition, such as gas, liquid, solid, and plasma.*

Two phases may coexist.

The curve of pressure versus volume for a matter at constant temperature, is called an isotherm, and is determined by the free energy of the substance.

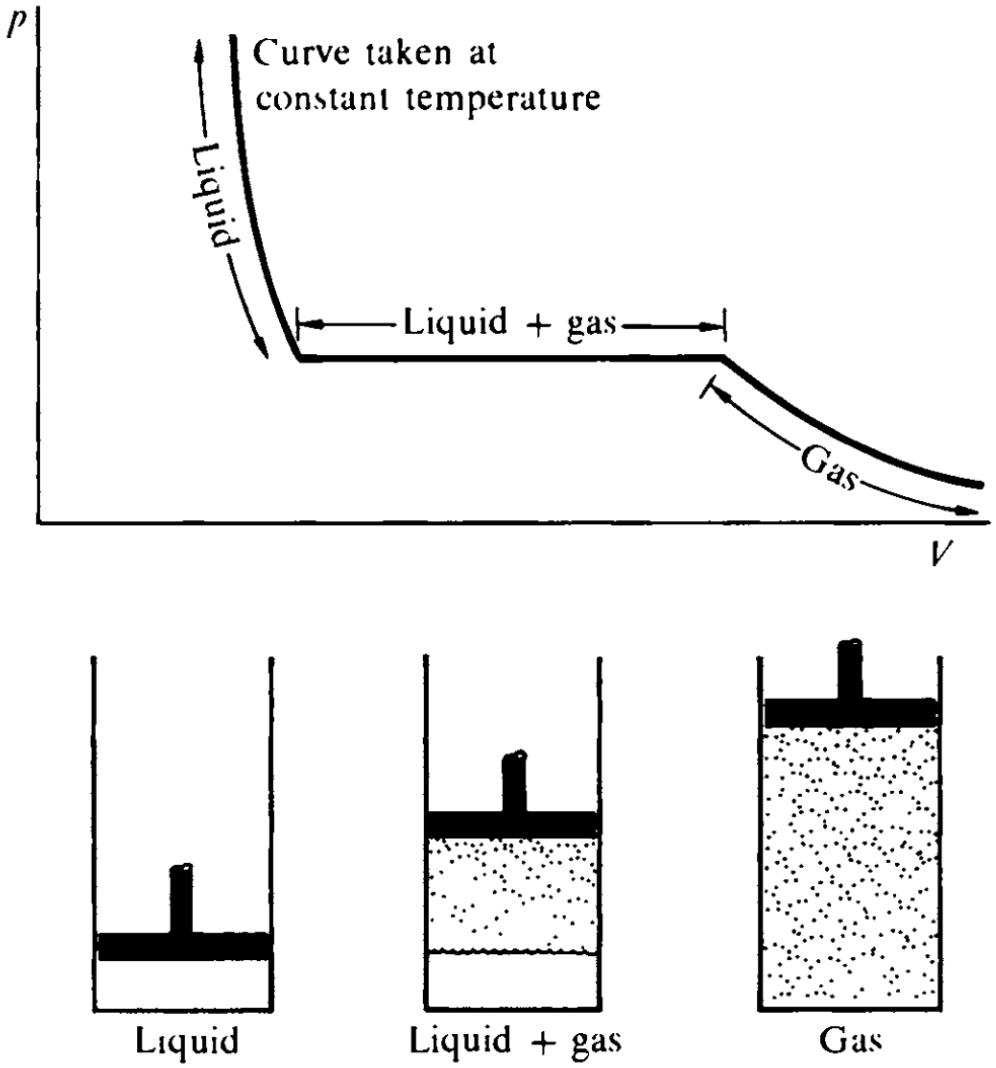

Figure 8.1: isotherm

An isotherm of a real gas may show a region in the p-V plane in which liquid and gas coexist in equilibrium with each other.

Liquid and gas may coexist on a section of an isotherm.

The thermodynamic conditions for the coexistence of two phases are the conditions for the equilibrium of two systems that are in thermal, diffusive, and mechanical contact. For liquid and gas, for example, the conditions are

$$\tau_l = \tau_g, \quad p_l = p_g, \quad \mu_l(p,\tau) = \mu_g(p,\tau).$$

If $\mu_l < \mu_g$ the liquid phase alone is stable, and if $\mu_g < \mu_l$ the gas phase alone is stable.

A triple point is where all three phases, gas, liquid, solid are in equilibrium, and thus the chemical potential$\mu_s = \mu_l = \mu_g$.

Metastable phases may occur, by supercooling or superheating.

$\mu_l(p,\tau) = \mu_g(p,\tau)$ gives us a differential relation of the coexistence curve of liquid and gas, the Clausius–Clapeyron relation:

$$\frac{dp}{d\tau} = \frac{s_g - s_l}{v_g - v_l} = \frac{L}{\tau \Delta v},$$

where N is the number of particles, $v = V/N, s = \sigma/N$, $s_g - s_l$ is the increase of entropy of the system when we transfer one molecule from the liquid to the gas, and $\Delta v = v_g - v_l$ is the increase of volume of the system when we transfer one molecule from the liquid to the gas.

The enthalpy

$$H = U + pV.$$

The latent heat of vaporization is the quantity of heat added in the transfer

$$L = dQ = \tau(s_g - s_l) = \tau\Delta\sigma = \Delta H = H_g - H_l.$$

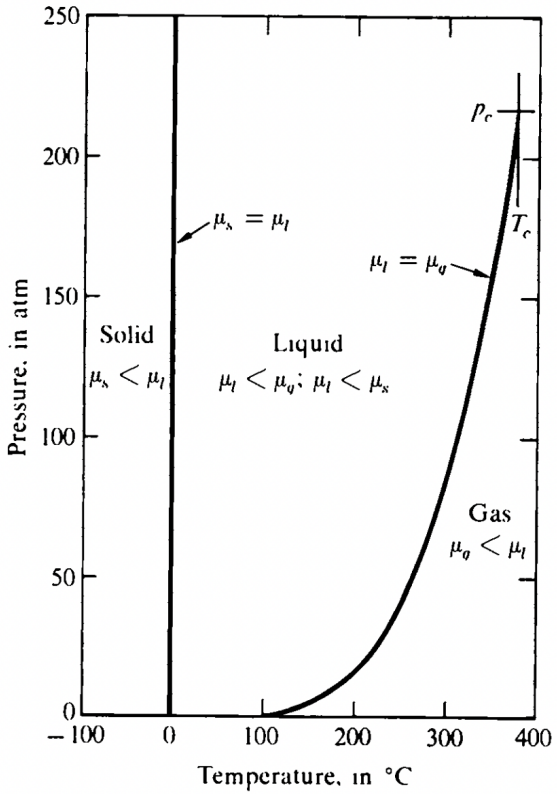

Figure 10.2 Phase diagram of H_2O. The relationships of the chemical potentials μ_s, μ_l, and μ_g in the solid, liquid, and gas phases are shown. The phase boundary here between ice and water is not exactly vertical; the slope is actually negative, although very large. After *International Critical Tables*, Vol. 3, and P. W. Bridgman, Proc. Am. Acad. Sci. **47**, 441 (1912); for the several forms of ice, see Zemansky, p. 375.

Figure 8.2: water

8.1 Mean Field Method

Example 96 (van der Walls equation of state)**.** *Take into account of the interactions between molecules, the ideal gas equation $pV = N\tau$ shall be modified.*

For ideal gas $F = -N\tau(\log(n_Q/n) + 1)$. Let b be the volume per molecule, the free volume is $V - Nb$, so the modified free energy to take into account of the short range repulsion is

$$F = -N\tau(\log(n_Q\frac{V - Nb}{N}) + 1).$$

Now we add a correction to the intermolecular attractive forces.

When the concentration of atoms in the gas is $n = N/V$, the average value of the total interaction of all other atoms on a particular atom can be approximated by the mean field method as $-2na$, where the factor 2 is used for convention. The interaction changes the

free energy in a gas of N molecules of volume V by $1/2 * N * -2na = -N^2a/V$. *So the modified free energy is:*

$$F = -N\tau(\log(n_Q\frac{V-Nb}{N}) + 1) - \frac{N^2a}{V}.$$

Since pressure $p = -\frac{\partial F}{\partial V}$, *we get the van der Walls equation of state:*

$$(p + N^2a/V^2)(V - Nb) = N\tau.$$

Definition 97 (critical point)**.** *The critical point in the context of the van der Waals equation of state represents a specific thermodynamic state where the distinction between liquid and gas phases disappears. At this point, the phase transition becomes continuous rather than first-order, and the fluid exhibits unique properties.*

Mathematically, the critical point is defined by the conditions where both the first and second derivatives of pressure with respect to volume at constant temperature vanish:

$$\left(\frac{\partial p}{\partial V}\right)_T = 0 \quad \textit{and} \quad \left(\frac{\partial^2 p}{\partial V^2}\right)_T = 0$$

These conditions lead to the critical parameters (p_c, V_c, T_c) *which are related to the van der Waals constants a and b as follows:*

$$V_c = 3Nb, \quad T_c = \frac{8a}{27b}, \quad p_c = \frac{a}{27b^2}$$

At temperatures above the critical temperature T_c, *the gas cannot be liquefied by pressure alone. Below* T_c, *the van der Waals equation predicts regions where* $\left(\frac{\partial p}{\partial V}\right)_T > 0$, *which are physically unstable and correspond to phase transitions in real systems.*

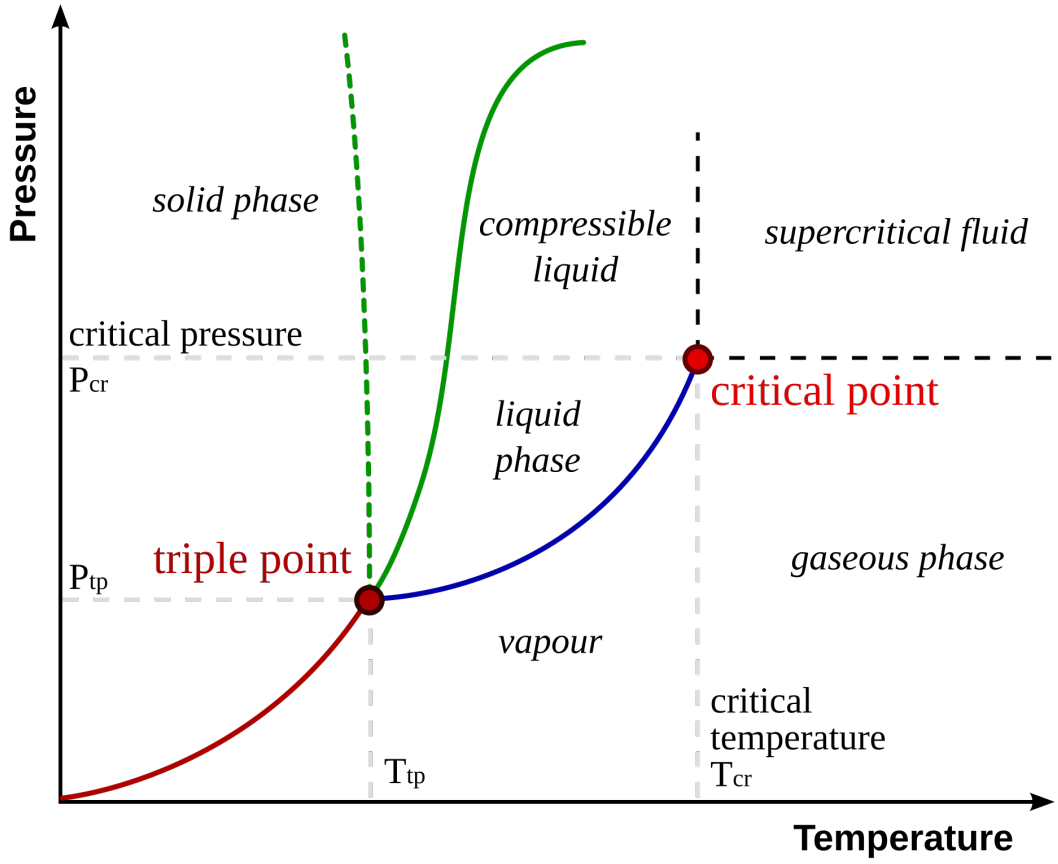

Figure 8.3: critical point

Example 98 (Ferromagnetism)**.** *Consider N magnets each with magnetic moment μ, with $N_+ = \frac{1}{2}N + s$ magnets up and $N_- = \frac{1}{2}N - s$ magnets down. The net magnetic moment is $2s\mu$. The potential energy of the system in a uniform magnetic field B is $U = -2s\mu \cdot B$. The entropy $\sigma(N, s)$ depends on the spin excess s and number of magnets N. The free energy is $F = U - \tau\sigma(N, s)$. At thermal equilibrium F is a minimum and $\partial F/\partial s = 0$, which gives*

$$< 2s >= N \tanh \mu B/\tau, \quad M =< 2s > \mu/V = n\mu \tanh \mu B/\tau, \quad Z = e^{\mu B/\tau} + e^{-\mu B/\tau} = 2\cosh \mu B\tau$$

A ferromagnet has a spontaneous magnetic moment, which means a magnetic moment even in zero applied magnetic field. The magnetization is defined as the magnetic moment per unit volume. The central assumption is that each magnetic atom experiences an effective field B_E proportional to the magnetization M: $B_E = \lambda M$, where λ is a constant.

So for a ferromagnet, $M = n\mu \tanh \mu\lambda M/\tau$, a transcendental equation for M.

Denote the reduced temperature $t = \tau/n\mu^2\lambda$, reduced magnetization $m = M/n\mu$, the equation becomes

$$m = \tanh(m/t)$$

The Curie temperature is the critical temperature $t = 1, \tau_c = n\mu^2\lambda$.

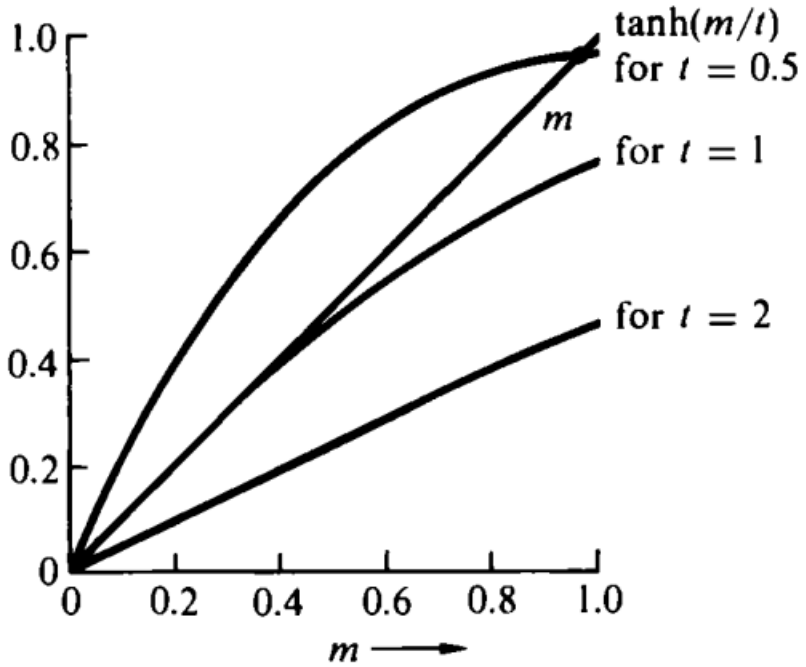

Figure 10.15 Graphical solution of Eq. (56) for the reduced magnetization m as a function of temperature. The reduced magnetization is defined as $m = M/n\mu$. The left-hand side of Eq. (56) is plotted as a straight line m with unit slope. The right-hand side is $\tanh(m/t)$ and is plotted versus m for three different values of the reduced temperature $t = \tau/n\mu^2\lambda = \tau/\tau_c$. The three curves correspond to the temperatures $2\tau_c$, τ_c, and $0.5\tau_c$. The curve for $t = 2$ intersects the straight line m only at $m = 0$, as appropriate for the paramagnetic region (there is no external applied magnetic field). The curve for $t = 1$ (or $\tau = \tau_c$) is tangent to the straight line m at the origin; this temperature marks the onset of ferromagnetism. The curve for $t = 0.5$ is in the ferromagnetic region and intersects the straight line m at about $m = 0.94\ n\mu$. As $t \to 0$ the intercept moves up to $m = 1$, so that all magnetic moments are lined up at absolute zero.

Figure 8.4: ferromagnetism

The critical point in the ferromagnetic system is the Curie temperature.

- ***Symmetry Breaking**: In the paramagnetic phase ($\tau > \tau_c$), spins are randomly oriented, and the system has rotational symmetry. Below the Curie temperature ($\tau < \tau_c$), this symmetry is spontaneously broken as spins align in a preferred direction.*

- *Diverging Susceptibility: The magnetic susceptibility χ diverges at τ_c, following $\chi \propto |\tau - \tau_c|^{-\gamma}$, indicating the system becomes extremely responsive to external fields.*

Definition 99 (order parameter, critical exponents, universality)**.** *An order parameter is a measure of the degree of order across the boundaries in a phase transition system; it normally ranges between zero in one phase (usually above the critical point) and nonzero in the other. At the critical point, the order parameter susceptibility will usually diverge.*

The control parameter that drives phase transitions is often temperature but can also be other macroscopic variables like pressure or an external magnetic field. For example, we want to describe the behavior of a physical quantity f in terms of a power law around the critical temperature; we introduce the reduced temperature

$$\tau := \frac{T - T_\mathrm{c}}{T_\mathrm{c}}$$

which is zero at the phase transition, and define the critical exponent k as:

$$k \stackrel{def}{=} \lim_{\tau \to 0} \frac{\log |f(\tau)|}{\log |\tau|}$$

This results in the power law we were looking for:

$$f(\tau) \propto \tau^k, \quad \tau \to 0$$

For example, magnetization is an order parameter in ferromagnetic transitions as the magnitude of magnetization directly measures the degree of alignment of the magnetic moments, representing how "ordered" the system is, and it can be described by a critical exponent.

- ***Zero to Non-Zero Transition**: Magnetization is precisely zero in the paramagnetic phase (when averaged over time or volume) and becomes non-zero in the ferromagnetic phase, following the definition of an order parameter.*

- ***Continuous Behavior**: Near the critical point, magnetization follows a power law: $M \propto (\tau_c - \tau)^\beta$, where β is a critical exponent, allowing for mathematical analysis of the transition.*

Theorem 100 (universality)**.** *TODO*

Example 101 (Landau Theory, second order phase transition)**.** *Assume the system can be described by an order parameter ξ and temperature τ. At thermal equilibrium $\xi(\tau) = \xi_0$. The Helmholtz free energy $F(\xi, \tau) = U(\xi, \tau) - \tau\sigma(\xi, \tau)$. Further assume F is an even function of ξ and perform a Taylor expansion, $F(\xi, \tau) = g_0(\tau) + \frac{1}{2} g_2(\tau)\xi^2 + \ldots$.*

A first order phase transition is characterized by a latent heat, such as the liquid-gas transition at constant pressure. The Laudau function that describes a first order transition has $g_2(\tau) = (\tau - \tau_0)\alpha, \quad \alpha > 0, \quad g_4 < 0, \quad g_6 > 0$, where τ_0 is the instability point, $g_2(\tau)$ changes sign at τ_0. Usually, the critical temperature $\tau_c > \tau_0$ because the system can transition before reaching the instability point.

Set $\partial F / \partial \xi = 0$, we get two roots $\xi = 0, \xi_2 \neq 0$ representing two local minima. At critical temperature τ_c, the two roots give the same free energy.

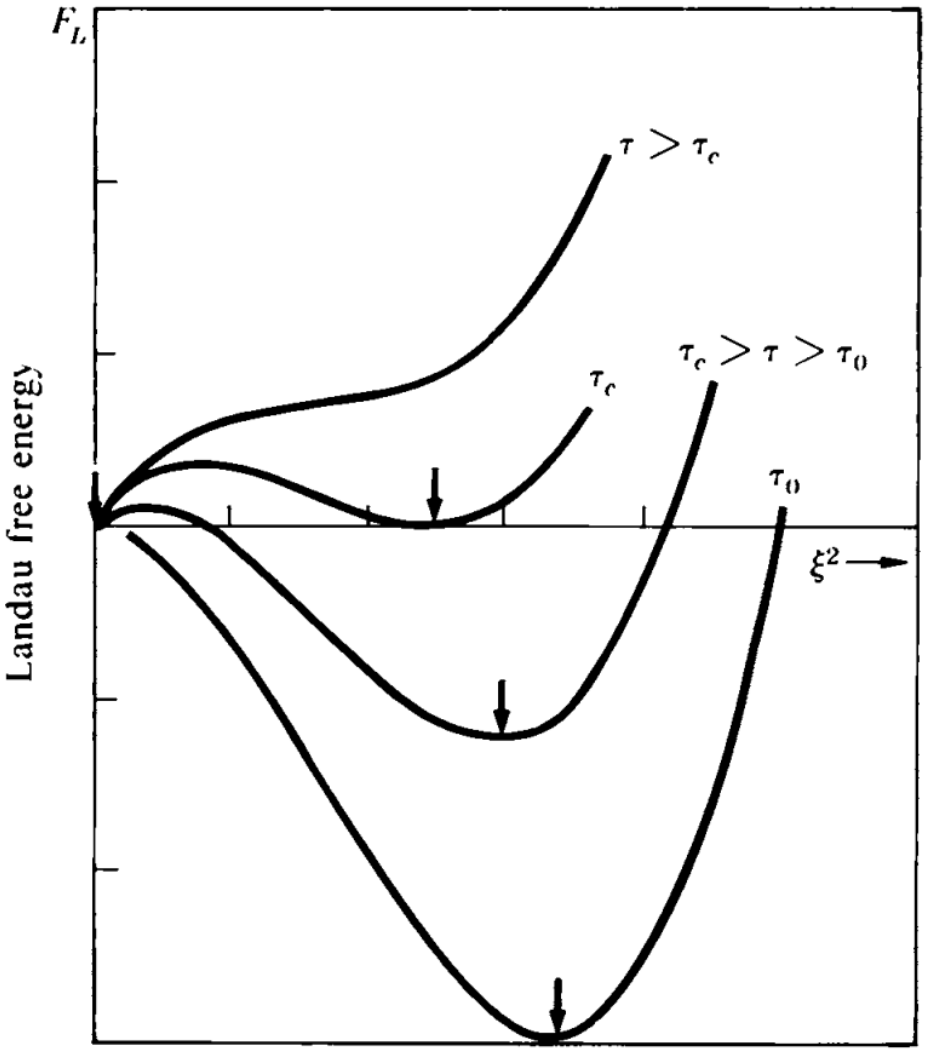

Figure 10.20 Landau free energy function versus ξ^2 in a first order transition, at representative temperatures. At τ_c the Landau function has equal minima at $\xi = 0$ and at a finite ξ as shown. For τ below τ_c the absolute minimum is at larger values of ξ; as τ passes through τ_c there is a discontinuous change in the position of the absolute minimum. The arrows mark the minima.

Figure 8.5: first order transition

The order parameter ξ does not go continuously to 0 as $\tau \to \tau_c$, since first-order transitions are driven by the competition between multiple local minima in the free energy.

A second order phase transition is characterized by:

- *Continuous change in the order parameter at the transition point*
- *Discontinuity in the second derivative of the free energy with respect to thermodynamic variables like temperature or pressure*

- *No latent heat exchange during the transition*
- *Divergence of susceptibility and correlation length at the critical point*
- *Critical slowing down of system dynamics near the transition*
- *Broken symmetry as the system crosses the critical point*

For example, a second order phase transition occurs when $g_2(\tau) = (\tau - \tau_0)\alpha, \quad \alpha > 0$ *and* $g_4 > 0$ *constant:*

$$F(\xi, \tau) = g_0(\tau) + \frac{1}{2}\alpha(\tau - \tau_0)\xi^2 + \frac{1}{4}g_4\xi^4$$

For a given temperature τ*, the order parameter* ξ *that minimizes* F *satisfy:*

$$F(\xi, \tau) = \begin{cases} g_0(\tau), \quad \xi = 0 & \text{when } \tau > \tau_0, \\ g_0(\tau) - \frac{\alpha^2}{4g_4}(\tau - \tau_0)^2, \quad \xi^2 = \frac{\alpha(\tau - \tau_0)}{g_4} & \text{when } \tau < \tau_0, \end{cases}$$

This model describes a phase transition in which the value of the order parameter ξ *goes continuously to zero as the temperature is increased to* τ_0*.*

There is no latent heat at the transition temperature τ_0 *since the entropy* $\sigma = -\frac{\partial F}{\partial \tau}$ *is continuous at* $\tau = \tau_0$*.*

For example, the free energy of ferromagnet is

$$F(M) = \text{constant} - \frac{1}{2}M^2(\lambda - \frac{\tau}{n\mu^2}) + \text{higher order terms},$$

$\tau_0 = n\mu^2\lambda = \tau_c$*, therefore in the mean field approximation ferromagnets satisfy the Laudau theory.*

8.2 Symmetry

TODO

Example 102 (Spontaneous Symmetry Breaking)**.**

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